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SOME INTEGRALS INVOLVING MACROBERT'S E-FUNCTION AND MEIJER'S G-FUNCTION

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ABSTRACT

In this paper some integrals involving the product of Macrobert's E-function, Meijer's G-function and an exponential function have been evaluated. The particular cases of the integrals lead to certain generalizations of the known results.

1. INTRODUCTION

The object of this paper is to evaluate some integrals involving the product of Macrobert's E-function, Meijer's G-function and an exponential function by expressing the G-function as Mellin-Barnes type integral and interchanging the order of integrations. The importance of the integrals lies in the fact that on specialising the parameters, the integrals yield many results some of which are known and the other are believed to be new.

2. The following formulae have been established:

$$(2\cdot1) \quad \int_0^\infty e^{-\alpha x} x^{\rho-1} E(\alpha_1, \alpha_2 :: \beta x)\, G_{p,q}^{m,n}\left(z x^{\delta/t} \middle| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) dx$$

$$= (2\pi)^{(1-t)(m+n-\frac{1}{2}p-\frac{1}{2}q)+\frac{1}{2}-\frac{1}{2}\delta}\, t^{\Sigma b_q - \Sigma a_p + \frac{1}{2}p - \frac{1}{2}q + 1}\, \delta^{\rho-\frac{1}{2}}\, \alpha^{-\rho}$$

$$\times \sum_{r=0}^{\infty} \frac{\Gamma(\alpha_1+r)\,\Gamma(\alpha_2+r)}{\lfloor r} \left(\frac{\beta-\alpha}{\beta\,\delta}\right)^r$$

$$\times G_{tp+2\delta,\, tq+\delta}^{tm,\, tn+2\delta}\left(z^t\, t^{t(p-q)} \left(\frac{\delta}{\alpha}\right)^\delta \middle| \begin{matrix} \Delta(\delta, 1-\alpha_1-\rho), \Delta(\delta, 1-\alpha_2-\rho), \Delta(t, a_1), \ldots, \Delta(t, a_p) \\ \Delta(t, b_1), \ldots, \Delta(t, b_q), \Delta(\delta, 1-\alpha_1-\alpha_2-\rho-r) \end{matrix}\right)$$

where δ and t are positive integers, $\Delta(t, \alpha)$ represents the set of parameters,

$$\frac{\alpha}{t}, \frac{\alpha+1}{t}, \ldots, \frac{\alpha+t-1}{t},$$

$$2(m+n)>p+q,\ |\arg z|<(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi,\ Re(\alpha)>0,\ Re(\beta)>\tfrac{1}{2}Re(\alpha),$$

$$Re(\rho+\alpha_i+\frac{\delta}{t}b_j)>0,\ i=1,2,\ j=1,2,\ldots,m.$$

$$(2\cdot2)\qquad \int_0^\infty e^{-\sigma x}x^{\rho-1}E(\alpha,\beta::\sigma x)\,E(\lambda,\mu::\zeta x)\,G^{m,n}_{p,q}\left(z\,x^{\delta/t}\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)dx$$
$$=(2\pi)^{(1-t)(m+n-\frac{1}{2}p-\frac{1}{2}q)+\frac{1}{2}-\frac{1}{2}\delta}\ t^{\Sigma b_q-\Sigma a_p+\frac{1}{2}p-\frac{1}{2}q+1}\ \delta^{\rho-\frac{1}{2}}\ \sigma^{-\rho}\,\Gamma(\beta)$$
$$\times\sum_{r=0}^{\infty}\frac{\Gamma(\alpha+r)}{\lfloor r}\,\delta^{-r}\sum_{u=0}^{\infty}\frac{\Gamma(\lambda+u)\,\Gamma(\mu+u+r)}{\lfloor u}\left(\frac{\zeta-\sigma}{\zeta\delta}\right)^u\ G^{tm,\,tn+3\delta}_{tp+3\delta,\,tq+2\delta}$$
$$\left(z^t\,t^{t(p-q)}\left(\frac{\delta}{\sigma}\right)^{\delta}\,\middle|\,\begin{matrix}\triangle(\delta,1-\alpha-\lambda-\rho),\ \triangle(\delta,1-\beta-\mu-\rho),\ \triangle(\delta,1-\alpha-\mu-\rho-r),\ \triangle(t,a_1),\ldots,\triangle(t,a_p)\\ \triangle(t,b_1),\ldots,\triangle(t,b_q),\ \triangle(\delta,1-\alpha-\beta-\mu-\rho-r),\ \triangle(\delta,1-\lambda-\mu-d-\rho-r-u)\end{matrix}\right)$$

where δ and t are positive integers,

$$2(m+n)>p+q,\ |\arg z|<(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi,\ Re(\sigma)>0,\ Re(\zeta)>\tfrac{1}{2}Re(\sigma),$$

$$Re(\rho+i_1+i_2+\frac{\delta}{t}b_j)>0,\ i_1=\alpha,\beta,\ i_2=\lambda,\mu,\ j=1,2,\ldots,m.$$

In the above integrals if we put $\alpha=\beta$ and $\sigma=\zeta$ respectively, we have their simplified forms as

$$(2\cdot3)\qquad \int_0^\infty e^{-\alpha x}x^{\rho-1}E(\alpha_1,\alpha_2::\alpha x)\,G^{m,n}_{p,q}\left(z\,x^{\delta/t}\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)dx$$
$$=(2\pi)^{(1-t)(m+n-\frac{1}{2}p-\frac{1}{2}q)+\frac{1}{2}-\frac{1}{2}\delta}\ t^{\Sigma b_q-\Sigma a_p+\frac{1}{2}p-\frac{1}{2}q+1}\ \delta^{\rho-\frac{1}{2}}\ \alpha^{-\rho}\,\Gamma(\alpha_1)\,\Gamma(\alpha_2)$$
$$\times G^{tm,\,tn+2\delta}_{tp+2\delta,\,tq+\delta}\left(z^t\,t^{t(p-q)}\left(\frac{\delta}{\alpha}\right)^{\delta}\,\middle|\,\begin{matrix}\triangle(\delta,1-\alpha_1-\rho),\ \triangle(\delta,1-\alpha_2-\rho),\ \triangle(t,a_1),\ldots,\triangle(t,a_p)\\ \triangle(t,b_1),\ldots,\triangle(t,b_q),\ \triangle(\delta,1-\alpha_1-\alpha_2-\rho)\end{matrix}\right)$$

where δ and t are positive integers,

$$2(m+n)>p+q,\ |\arg z|<(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi,\ Re(\alpha)>0,$$
$$Re(\rho+\alpha_i+\frac{\delta}{t}b_j)>0,\ i=1,2,\ j=1,2,\ldots,m.$$

$$(2\cdot4)\qquad \int_0^\infty e^{-\sigma x}x^{\rho-1}E(\alpha,\beta::\sigma x)\,E(\lambda,\mu::\sigma x)\,G^{m,n}_{p,q}\left(z\,x^{\delta/t}\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)dx$$
$$=(2\pi)^{(1-t)(m+n-\frac{1}{2}p-\frac{1}{2}q)+\frac{1}{2}-\frac{1}{2}\delta}\ t^{\Sigma b_q-\Sigma a_p+\frac{1}{2}p-\frac{1}{2}q+1}\ \delta^{\rho-\frac{1}{2}}\ \sigma^{-\rho}\,\Gamma(\beta)\,\Gamma(\lambda)$$
$$\times\sum_{r=0}^{\infty}\frac{\Gamma(\alpha+r)\Gamma(\mu+r)}{\lfloor r}\,\delta^{-r}\ G^{tm,\,tn+3\delta}_{tp+3\delta,\,tq+2\delta}$$
$$\left(z^t\,t^{t(p-q)}\left(\frac{\delta}{\sigma}\right)^{\delta}\,\middle|\,\begin{matrix}\triangle(\delta,1-\alpha-\lambda-\rho),\ \triangle(\delta,1-\beta-\mu-\rho),\ \triangle(\delta,1-\alpha-\mu-\rho-r),\ \triangle(t,a_1),\ldots,\triangle(t,a_p)\\ \triangle(t,b_1),\ldots,\triangle(t,b_q),\ \triangle(\delta,1-\alpha-\beta-\mu-\rho-r),\ \triangle(\delta,1-\lambda-\mu-\alpha-\rho-r)\end{matrix}\right)$$

where δ and t are positive integers,

$$2(m+n)>p+q,\ |\arg z|<(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi,\ Re(\sigma)>0,$$
$$Re(\rho+i_1+i_2+\frac{\delta}{t}b_j)>0,\ i_1=\alpha,\beta,\ i_2=\lambda,\mu,\ j=1,2,\ldots,m.$$

Proof :

In the beginning consider the integral (2·1) with $t=1$. To establish the integral expressing the G-function in the integrand as Mellin-Barnes type integral [1, p. 207, (1)], and interchanging the order of integrations, which is justifiable due to the absolute convergence of the integrals involved in the process, we have

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\, Z^s \int_0^\infty e^{-\alpha x}\, x^{\rho+\delta s-1}\, E(\alpha_1, \alpha_2 :: \beta x)dx\, ds$$

Now evaluating the inner integral with the help of the results

$$\int_0^\infty e^{-\alpha x}\, x^{\rho-1}\, E(\alpha_1, \alpha_2 :: \beta x)dx = \alpha^{-\rho}\,\Gamma(\alpha_1+\rho)\,\Gamma(\alpha_2+\rho)\sum_{r=0}^{\infty}\frac{\Gamma(\alpha_1+r)\,\Gamma(\alpha_2+r)}{\lfloor r\,\Gamma(\alpha_1+\alpha_2+\rho+r)}\left(\frac{\beta-\alpha}{\beta}\right)^r$$

$Re(\alpha)>0$, $Re(\beta)>\frac{1}{2}\, Re(\alpha)$, $Re(\alpha_1+\rho) > 0$, $Re(\alpha_2+\rho)>0$

which follows from [2, p. 396, (113)], and using the multiplication formula for gamma-function, we have

$$\alpha^{-\rho}\,(2\pi)^{\frac{1}{2}-\frac{1}{2}\delta}\,\delta^{\rho-\frac{1}{2}}\sum_{r=0}^{\infty}\frac{\Gamma(\alpha_1+r)\,\Gamma(\alpha_2+r)}{\lfloor r}\left(\frac{\beta-\alpha}{\beta\delta}\right)^r$$

$$\times\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)\prod_{k=0}^{\delta-1}\Gamma\left(\frac{\alpha_1+\rho+k}{\delta}+s\right)\prod_{k=0}^{\delta-1}\Gamma\left(\frac{\alpha_2+\rho+k}{\delta}+s\right)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)\prod_{k=0}^{\delta-1}\Gamma\left(\frac{\alpha_1+\alpha_2+\rho+r+k}{\delta}+s\right)}$$

$$\times\, Z^s\left(\frac{\delta}{\alpha}\right)^{\delta s} ds$$

On applying [1, p. 207, (1)], the value of the integral (2·1) with $t=1$ is obtained as

$$\alpha^{-\rho}\,(2\pi)^{\frac{1}{2}-\frac{1}{2}\delta}\,\delta^{\rho-\frac{1}{2}}\sum_{r=0}^{\infty}\frac{\Gamma(\alpha_1+r)\,\Gamma(\alpha_2+r)}{\lfloor r}\left(\frac{\beta-\alpha}{\beta\delta}\right)^{\delta}$$

$$\times\, G^{m,\,n+2\delta}_{p+2\delta,\,q+\delta}\left(z\left(\frac{\delta}{\alpha}\right)^{\delta}\,\middle|\,\begin{matrix}\triangle(\delta,\,1-\alpha_1-\rho),\ \triangle(\delta,\,1-\alpha_2-\rho),\ a_1,\ldots,a_p\\ b_1,\ldots,b_q,\ \triangle(\delta,\,1-\alpha_1-\alpha_2-\rho-r)\end{matrix}\right)$$

By virtue of the relation [5, p. 39], *viz* :

$$G^{m,\,n}_{p,\,q}\left(zx^{\delta/t}\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right) = (2\pi)^{(1-t)(m+n-\frac{1}{2}p-\frac{1}{2}q)}\, t^{\Sigma b_q-\Sigma a_p+\frac{1}{2}p-\frac{1}{2}q+1}$$

$$\times\, G^{tm,\,tn}_{tp,\,tq}\left(z^t\, t^{t(p-q)}\, x^{\delta}\,\middle|\,\begin{matrix}\triangle(t,\,a_1),\ldots,\triangle(t,\,a_p)\\ \triangle(t,\,b_1),\ldots,\triangle(t,\,b_q)\end{matrix}\right)$$

where δ and t are positive integers,

the integral established above with $t=1$ in (2·1) can be reduced to the formula (2·1).

The formula (2·2) can be similarly established on applying the same procedure as above and using the result

$$\int_0^\infty e^{-\sigma x} x^{\rho-1} E(\alpha, \beta :: \sigma x) E(\lambda, \mu :: \zeta x)\, dx = \sigma^{-\rho} \Gamma(\beta)\Gamma(\alpha+\rho+\lambda)\Gamma(\beta+\rho+\mu)$$

$$\times \sum_{r=0}^{\infty} \frac{\Gamma(\alpha+r)\Gamma(\alpha+\rho+\mu+r)}{|\underline{r}\ \Gamma(\alpha+\beta+\rho+\mu+r)} \sum_{u=0}^{\infty} \frac{\Gamma(\lambda+u)\Gamma(\mu+r+u)}{|\underline{u}\ \Gamma(\alpha+\rho+\lambda+\mu+r+u)} \left(\frac{\zeta-\rho}{\zeta}\right)^u$$

$Re\,(\sigma)>0$, $Re\,(\zeta)>\frac{1}{2}\,Re\,(\sigma)$, $Re\,(\alpha+\rho+\lambda)>0$, $Re\,(\beta+\rho+\lambda)>0$,

$Re\,(\alpha+\rho+\mu)>0$, $Re\,(\beta+\rho+\lambda)>0$,

which follows from [3, p. 397, (117)]

3. *Particular Cases.*

By specializing the parameters, the G-function and the E-functions involved in the integrals can be reduced to many simple functions. However the following interesting cases, some of which give the generalizations of certain known results are worth mentioning.

3·1. Particular cases of the integral (2·1) involving E-function.

(*i*) In the left hand side of (2·1) reducing the G-function to E-function with the help of [2, p. 444, (2)], we get

$$(3\cdot1)\qquad \int_0^\infty e^{-\alpha x} x^{\rho-1} E(\alpha_1, \alpha_2 :: \beta x)\ E\left(p\,;\, a_r : q\,;\, b_s :: z x^{\delta/t}\right) dx$$

$$= (2\pi)^{(1-t)(\frac{1}{2}p-\frac{1}{2}q+\frac{1}{2})+\frac{1}{2}-\frac{1}{2}\delta}\ t^{\Sigma a_p-\Sigma b_q+\frac{1}{2}q-\frac{1}{2}p+\frac{1}{2}}\ \delta^{\rho-\frac{1}{2}}\ \alpha^{-\rho}$$

$$\times \sum_{r=0}^{\infty} \frac{\Gamma(\alpha_1+r)\Gamma(\alpha_2+r)}{|\underline{r}} \left(\frac{\beta-\alpha}{\beta\delta}\right)^r G^{tp,\ t+2\delta}_{t(q+1)-2\delta,\ tp+\delta}$$

$$\left(z^t\ t^{t(q-p+1)} \left(\frac{\delta}{\alpha}\right)^\delta \,\middle|\, \begin{matrix} \Delta(\delta, 1-\alpha_1-\rho), \Delta(\delta, 1-\alpha_2-\rho), \Delta(t,1), \Delta(t, b_1), \ldots, \Delta(t, b_q) \\ \Delta(t, a_1), \ldots, \Delta(t, a_p), \Delta(\delta, 1-\alpha_1-\alpha_2-\rho-r) \end{matrix}\right)$$

where δ and t are positive integers,

$p+1>q$, $|\arg z| < \frac{1}{2}(p-q+1)\pi$, $Re\,(\alpha)>0$, $Re\,(\beta)>\frac{1}{2}\,Re(\alpha)$,

$Re\,(\rho+\alpha_i+\frac{\delta}{t}\,a_j)>0$, $i=1, 2$, $j=1, 2, \ldots, p$.

In (3·1), if we take $\alpha=\beta=1$, $\delta=2$, $t=1$ and use the result [1, p. 208, (6)] it reduces to the known result obtained by Ragab [3, p. 409, (42)].

In (3·1), putting $\delta=t=1$, $\alpha=\beta=b$, $\alpha_1=\frac{1}{2}-l-n$, $\alpha_2=\frac{1}{2}-l+n$, $a_1=\lambda$, $a_2=\alpha+m$, $a_3=\alpha-m$, $b_1=\frac{1}{2}+\alpha-k$ replacing ρ by $\rho+l-\lambda+1$, using the results [1, p. 209, (9)] and [1, p. 109, (8)] it reduces to the known result obtained by Saxena [6, p. 352, (51)].

(*ii*) In (2·1), if we put $m=1, n=p, b_1=0$, replace q by $q+1$, write $1-a_j$ and $1-b_{k-1}$ for a_j and b_k $(j=1, 2, \ldots, p)$ and $(k=2, 3, \ldots, q+1)$, then by virtue of [1, p. 209, (9)] and [2, p. 444, (2)], we have

$$(3{\cdot}2)\quad \int_0^\infty e^{-\alpha x}\; x^{\rho-1}\, E(\alpha_1, \alpha_2 :: \beta x)\, E\left(p; a_r : q; b_s :: \frac{1}{z}\, x^{-\delta/t}\right) dx$$

$$=(2\pi)^{(1-t)(\frac{1}{2}p-\frac{1}{2}q+\frac{1}{2})+\frac{1}{2}-\frac{1}{2}\delta}\; t^{\Sigma a_p-\Sigma b_q+\frac{1}{2}(q-p)+\frac{1}{2}}\; \delta^{\rho-\frac{1}{2}}\; \alpha^{-\rho}$$

$$\sum_{r=0}^{\infty} \frac{\Gamma(\alpha_1+r)\,\Gamma(\alpha_2+r)}{\lfloor r} \left(\frac{\beta-\alpha}{\beta\delta}\right)^r G^{t,\, tp+2\delta}_{tp+2\delta,\, t(q+1)+\delta}$$

$$\left(z^t\, t^{t(p-q-1)}\left(\frac{\delta}{\alpha}\right)^\delta \middle| \begin{matrix} \Delta(\delta, 1-\alpha_1-\rho),\, \Delta(\delta, 1-\alpha_2-\rho),\, \Delta(t, 1-a_1), \ldots, \Delta(t, 1-a_p) \\ \Delta(t, o),\, \Delta(t, 1-b_1), \ldots, \Delta(t, 1-b_q),\, \Delta(\delta, 1-\alpha_1-\alpha_2-\rho-r) \end{matrix}\right)$$

where δ and t are positive integers,

$p+1>q$, $|\arg z| <\frac{1}{2}(p-q+1)\pi$, $Re\,(\alpha)>0$, $Re\,(\beta)>\frac{1}{2}\, Re\,(\alpha)$, $Re\,(\rho+\alpha_i)>0$, $i=1, 2$.

In (3·2) if we take $\alpha=\beta=1$, $t=1$, it reduces to the known result obtained by Macrobert [4, p. 191, (11)].

3·2. Particular cases of the integral (2·1) involving Whittaker functions.

If we take $\alpha_1=\frac{1}{2}-k+\mu$, $\alpha_2=\frac{1}{2}-k-\mu$ and use the relation [3, p. 351, (15)], then (2·1) is reduced to the form

$$(3{\cdot}3)\quad \int_0^\infty e^{-\alpha x+\frac{1}{2}\beta x}\, x^{\rho-1}\; W_{k,\mu}(\beta x)\; G^{m,n}_{p,q}\left(z\, x^{\delta/t} \middle| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) dx$$

$$=(2\pi)^{(1-t)(m+n-\frac{1}{2}p-\frac{1}{2}q)+\frac{1}{2}-\frac{1}{2}\delta}\; t^{\Sigma b_q-\Sigma a_p+\frac{1}{2}p-\frac{1}{2}q+1}\; \delta^{\rho+k-\frac{1}{2}}$$

$$\times \frac{\alpha^{-\rho-k}\,\beta^k}{\Gamma(\frac{1}{2}-k+\mu)\Gamma(\frac{1}{2}-k-\mu)} \sum_{r=0}^{\infty} \frac{\Gamma(\frac{1}{2}-k+\mu+r)\,\Gamma(\frac{1}{2}-k-\mu+r)}{\lfloor r} \left(\frac{\beta-\alpha}{\beta\delta}\right)^r$$

$$\times\; G^{tm,\, tn+2\delta}_{tp+2\delta,\, tq+\delta}\left(z^t\, t^{t(p-q)}\left(\frac{\delta}{\alpha}\right)^\delta \middle| \begin{matrix} \Delta(\delta, \frac{1}{2}+\mu-\rho),\, \Delta(\delta, \frac{1}{2}-\mu-\rho),\, \Delta(t, a_1), \ldots, \Delta(t, a_p) \\ \Delta(t, b_1), \ldots, \Delta(t, b_q),\, \Delta(\delta, k-\rho-r) \end{matrix}\right)$$

where δ and t are positive integers,

$2(m+n)>(p+q)$, $|\arg z| <(m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$, $Re\,(\alpha)>0$, $Re\,(\beta)>\frac{1}{2}\, Re\,(\alpha)$,

$Re\,(\rho+\frac{1}{2}\pm\mu+\frac{\delta}{t}\, b_j)> 0$, $j=1, 2, \ldots, m$.

It we take $\alpha=\beta$, (3·3) reduces to a known result obtained by Saxena [7, p. 402, (10)].

In (3·3) taking $\alpha=\beta=1$, $t=1$, $m=1$, $n=p$, $b_1=0$ replacing q by $q+1$, writing $1-a_j$ and $1-b_{k-1}$ for a_j and b_k $(j=1, 2, \ldots, p)$ and $(k=2, 3, \ldots, q+1)$ then applying [1, p. 209, (9)] and [2, p. 444, (2)] it reduces to a known result [2, p. 416, (10)].

Reducing the G-function in (3·3) to a Whittaker function we get the following six integrals, (3·4) to (3·9).

(*i*) In (3·3) taking $m=2$, $n=1$, $p=1$, $q=2$, $a_1=\frac{1}{2}+\lambda-\nu$, $b_1=0$, $b_2=-2\nu$ and using the result [2, p. 435, (5)] it reduces to the form

$$(3{\cdot}4)\qquad \int_0^\infty e^{-\alpha x+\frac{1}{2}\beta x+\frac{1}{2}zx^{\delta/t}}\, x^{\rho-1}\, W_{k,\mu}(\beta x)\, W_{\lambda,\nu}(zx^{\delta/t})\,dx$$

$$=(2\pi)^{\frac{\delta}{2}(1-t)+\frac{1}{2}x-\frac{1}{2}\delta}\; t^{-\lambda-\nu}\; \delta^{k+\rho+\frac{\delta}{t}(\frac{1}{2}+\nu)-\frac{1}{2}}\; \alpha^{-\rho-\frac{\delta}{t}(\frac{1}{2}+\nu)-k}\beta^k\; z^{\frac{1}{2}+\nu}$$

$$\times\frac{1}{\Gamma(\frac{1}{2}-\lambda+\nu)\,\Gamma(\frac{1}{2}-\lambda-\nu)\,\Gamma(\frac{1}{2}-k+\mu)\,\Gamma(\frac{1}{2}-k-\mu)}\sum_{r=0}^\infty \frac{\Gamma(\frac{1}{2}-k+\mu+r)\,\Gamma(\frac{1}{2}-k-\mu+r)}{\lfloor r}\left(\frac{\beta-\alpha}{\beta\delta}\right)^r$$

$$\times\; G^{2t,\,t+2\delta}_{t+2\delta,\,2t+\delta}\left(\frac{z^t\,\delta^\delta}{t^t\,\alpha^\delta}\,\middle|\,\begin{matrix}\Delta(\delta,\frac{1}{2}-\mu-\frac{\delta}{t}(\frac{1}{2}+\nu)-\rho),\ \Delta(\delta,\frac{1}{2}+\mu-\frac{\delta}{t}(\frac{1}{2}+\nu)-\rho),\ \Delta(t,\frac{1}{2}+\lambda-\nu)\\ \Delta(t,0),\ \Delta(t,-2\nu),\ \Delta(\delta,k-\frac{\delta}{t}(\frac{1}{2}+\nu)-\rho-r)\end{matrix}\right)$$

where δ and t are positive integers,

$$|\arg z|<\tfrac{3}{2}\pi,\ Re(\alpha)>0,\ Re(\beta)>\tfrac{1}{2}Re(\alpha),\ Re(\rho+\tfrac{\delta}{t}(\tfrac{1}{2}\pm\nu)+\tfrac{1}{2}\pm\mu)>0.$$

If we take $\alpha=\beta$, $\delta=t=1$, (3·4) reduces to the known result [2, p. 411, (45)].

(*ii*) In (3·3) putting $m=2$, $n=0$, $p=1$, $q=2$, $a_1=\frac{1}{2}-\lambda-\nu$, $b_1=0$, $b_2=-2\nu$ and using the result [2, p. 435, (3)] it reduces to the form

$$(3{\cdot}5)\qquad \int_0^\infty e^{-\alpha x+\frac{1}{2}\beta x-\frac{1}{2}zx^{\delta/t}}\, x^{\rho-1}\, W_{k,\mu}(\beta x)\, W_{\lambda,\nu}(zx^{\delta/t})\,dx$$

$$=(2\pi)^{\frac{1}{2}(t-1)+\frac{1}{2}-\frac{1}{2}\delta}\; t^{\lambda-\nu}\; \delta^{k+\rho+\frac{\delta}{t}(\frac{1}{2}+\nu)-\frac{1}{2}}\; \alpha^{-\rho-\frac{\delta}{t}(\frac{1}{2}+\nu)-k}\,\beta^k\; z^{\frac{1}{2}+\nu}$$

$$\times\frac{1}{\Gamma(\frac{1}{2}-k+\mu)\,\Gamma(\frac{1}{2}-k-\mu)}\sum_{r=0}^\infty \frac{\Gamma(\frac{1}{2}-k+\mu+r)\,\Gamma(\frac{1}{2}-k-\mu+r)}{\lfloor r}\left(\frac{\beta-\alpha}{\beta\delta}\right)^r$$

$$\times\; G^{2t,\,2\delta}_{t+2\delta,\,2t+\delta}\left(\frac{z^t\delta^\delta}{t^t\alpha^\delta}\,\middle|\,\begin{matrix}\Delta(\delta,\frac{1}{2}+\mu-\frac{\delta}{t}(\frac{1}{2}+\nu)-\rho),\ \Delta(\delta,\frac{1}{2}-\mu-\frac{\delta}{t}(\frac{1}{2}+\nu)-\rho),\ \Delta(t,\frac{1}{2}-\lambda-\nu)\\ \Delta(t,0),\ \Delta(t,-2\nu),\ \Delta(\delta,k-\frac{\delta}{t}(\frac{1}{2}+\nu)-\rho-r)\end{matrix}\right)$$

where δ and t are positive integers,

$$|\arg z|<\pi/2,\ Re(\alpha)>0,\ Re(\beta)>\tfrac{1}{2}Re(\alpha),\ Re(\rho+\tfrac{\delta}{t}(\tfrac{1}{2}\pm\nu)+\tfrac{1}{2}\pm\mu)>0.$$

If we take $\alpha=\beta$ in (3·5) it reduces to a known result obtained by Saxena [7, p. 403, (12)].

Further with $\delta=t=1$ (3·5) yields a known result [2, p. 411, (46)].

(*iii*) In (3·3) with $m=1$, $n=1$, $p=1$, $q=2$, $a_1=\frac{1}{2}-\lambda-\nu$, $b_1=0$, $b_2=-2\nu$ and using the result [2, p. 442, (7)], it reduces to the form

$$(3{\cdot}6)\qquad \int_0^\infty e^{-\alpha x+\frac{1}{2}\beta x-\frac{1}{2}zx^{\delta/t}}\, x^{\rho-1}\, W_{k,\mu}(\beta x)\, M_{\lambda,\nu}(zx^{\delta/t})\,dx$$

$$=(2\pi)^{\frac{1}{2}(1-t)+\frac{1}{2}-\frac{1}{2}\delta}\; t^{\lambda-\nu}\; \delta^{\rho+k+\frac{\delta}{t}(\frac{1}{2}+\nu)-\frac{1}{2}}\alpha^{-\rho-\frac{\delta}{t}(\frac{1}{2}+\nu)-k}\; \beta^k\, z^{\frac{1}{2}+\nu}$$

$$\times\frac{\Gamma(1+2\nu)}{\Gamma(\frac{1}{2}-k+\mu)\Gamma(\frac{1}{2}-k-\mu)\Gamma(\frac{1}{2}+\lambda+\nu)}\sum_{r=0}^\infty\frac{\Gamma(\frac{1}{2}-k+\mu+r)\Gamma(\frac{1}{2}-k-\mu+r)}{\lfloor r}\left(\frac{\beta-\alpha}{\beta\delta}\right)^r$$

$$\times\; G^{2t,\,t+2\delta}_{t+2\delta,\,2t+\delta}\left(\frac{z^t\,\delta^\delta}{t^t\,\alpha^\delta}\,\middle|\,\begin{matrix}\Delta(\delta,\frac{1}{2}+\mu-\frac{\delta}{t}(\frac{1}{2}+\nu)-\rho),\ \Delta(\delta,\frac{1}{2}-\mu-\frac{\delta}{t}(\frac{1}{2}\nu)-\rho),\ (t,\frac{1}{2}-\lambda-\nu)\\ \Delta(t,0),\ \Delta(t,-2\nu),\ \Delta(\delta,k-\frac{\delta}{t}(\frac{1}{2}+\nu)-\rho-r)\end{matrix}\right)$$

where δ and t are positive integers,

$$|\arg z| < \frac{\pi}{2},\ Re(\alpha)>0,\ Re(\beta)>\tfrac{1}{2}\,Re(\alpha),\ Re(\rho+\tfrac{\delta}{t}(\tfrac{1}{2}\pm\nu)+\tfrac{1}{2}\pm\mu)>0$$

If we take $\alpha=\beta$, $\delta=t=1$, (3·6) yields the known result [2, p. 410, (43)]

(*iv*) In (3·3) taking $m=1$, $n=2$, $p=2$, $q=1$, $a_1=1$, $a_2=1+2\nu$, $b_1=\tfrac{1}{2}-\lambda+\nu$ using [1, p. 209, (9)] and [2, p. 435, (5)] it reduces to the form

$$(3\cdot7)\qquad \int_0^\infty e^{-\alpha x+\frac{1}{2}\beta x+\frac{1}{2}z^{x^{-\delta/t}}}\, x^{\rho-1}\, W_{k,\mu}(\beta x)\, W_{\lambda,\nu}\left(\frac{x^{-\delta/t}}{z}\right) dx$$

$$=(2\pi)^{\frac{1}{2}(1-t)+\frac{1}{2}-\frac{1}{2}\delta}\; t^{-\lambda-\nu}\; \delta^{k+\rho-\frac{\delta}{t}(\frac{1}{2}+\nu)-\frac{1}{2}}\; \alpha^{-\rho+\frac{\delta}{t}(\frac{1}{2}+\nu)-k}\; \beta^{k}\; z^{-\frac{1}{2}-\nu}$$

$$\times\frac{1}{\Gamma(\frac{1}{2}-k+\mu)\Gamma(\frac{1}{2}-k-\mu)\Gamma(\frac{1}{2}-\lambda+\nu)\Gamma(\frac{1}{2}-\lambda-\nu)}\sum_{r=0}^{\infty}\frac{\Gamma(\frac{1}{2}-k+\mu+r)\Gamma(\frac{1}{2}-k-\mu+r)}{\lfloor r}\left(\frac{\beta-\alpha}{\beta\delta}\right)^{r}$$

$$\times\, G^{t,2t+2\delta}_{2t+2\delta,t+\delta}\left((zt)^{t}\left(\frac{\delta}{\alpha}\right)^{\delta}\,\middle|\,\begin{matrix}\Delta(\delta,\frac{1}{2}+\mu+\frac{\delta}{t}(\frac{1}{2}+\nu)-\rho),\ \Delta(\delta,\frac{1}{2}-\mu+\frac{\delta}{t}(\frac{1}{2}+\nu)-\rho),\ \Delta(t,1),\ \Delta(t,1+2\nu)\\ \Delta(t,\frac{1}{2}-\lambda+\nu),\ \Delta(\delta,k-\rho+\frac{\delta}{t}(\frac{1}{2}+\nu)-r)\end{matrix}\right)$$

where δ and t are positive integers,

$$|\arg z| < \tfrac{3}{2}\pi,\ Re(\alpha)>0,\ Re(\beta)>\tfrac{1}{2}\,Re(\alpha),\ Re(\rho+\tfrac{1}{2}\pm\mu-\tfrac{\delta}{t}\lambda)>0.$$

Taking $\alpha=\beta$, $\delta=t=1$ and replacing x by $1/x$ in (3·7), it yields the known result [2, p. 412, (55)].

(*v*) Using [1, p. 209, (9)] in (3·3), then putting $m=0$, $n=2$, $p=2$, $q=1$, $a_1=1$, $a_2=1+2\nu$, $b_1=\tfrac{1}{2}+\lambda+\nu$ using [2, p. 435, (3)], it reduces to the form

$$(3\cdot8)\qquad \int_0^\infty e^{-\alpha x+\frac{1}{2}\beta x-\frac{1}{2}z\,x^{-\delta/t}}\, x^{\rho-1}\, W_{k,\mu}(\beta x)\, W_{\lambda,\nu}\left(\frac{x^{-\delta/t}}{z}\right) dx$$

$$=(2\pi)^{\frac{1}{2}(1-t)+\frac{1}{2}-\frac{1}{2}\delta}\; t^{\lambda-\nu}\; \delta^{k+\rho-\frac{\delta}{t}(\frac{1}{2}+\nu)-\frac{1}{2}}\; \alpha^{-\rho+\frac{\delta}{t}(\frac{1}{2}+\nu)-k}\; \beta^{k}\; z^{-\frac{1}{2}-\nu}$$

$$\times\frac{1}{\Gamma(\frac{1}{2}-k+\mu)\Gamma(\frac{1}{2}-k-\mu)}\sum_{r=0}^{\infty}\frac{\Gamma(\frac{1}{2}-k+\mu+r)\,\Gamma(\frac{1}{2}-k-\mu+r)}{\lfloor r}\left(\frac{\beta-\alpha}{\beta\delta}\right)^{r}$$

$$\times\, G^{2t+2\delta,0}_{t+\delta,2t+2\delta}\left(\frac{\alpha\delta}{z^{t}\,t^{t}\,\delta\delta}\,\middle|\,\begin{matrix}\Delta(t,\frac{1}{2}-\lambda-\nu),\ \Delta(\delta,1-k+\rho-\frac{\delta}{t}(\frac{1}{2}+\nu)+r)\\ \Delta(\delta,\frac{1}{2}-\mu+\rho-\frac{\delta}{t}(\frac{1}{2}+\nu),\ \Delta(\delta,\frac{1}{2}+\mu+\rho-\delta/t\,\frac{1}{2}+\nu),\Delta(t,0),\Delta(t,-2\nu)\end{matrix}\right)$$

where δ and t are positive integers,

$$|\arg z| < \frac{\pi}{2},\ Re\,(\alpha>0,\ Re(\beta)>\tfrac{1}{2}\,Re(\alpha),\ Re(\rho+\tfrac{1}{2}\pm\mu+\tfrac{\delta}{t}\lambda)>0.$$

If we take $\alpha=\beta$ in (3·8) it reduces to a known result obtained by Saxena [7, p. 404 (14)].

Further with $\delta=t=1$, (3·8) yields a known result [2, p. 412, (54)].

(*vi*) Finally on putting $m=1$, $n=1$, $p=2$, $q=1$, $a_1=1$, $a_2=1+2\nu$, $b_1=\frac{1}{2}+\lambda+\nu$ in (3·3), then using [1, p. 209 (9)], and [2, p. 442, (7)] it reduces to the form

$$(3\cdot 9)\qquad \int_0^\infty e^{-\alpha x+\frac{1}{2}\beta x-\frac{1}{2z}x^{-\delta/t}}\, x^{\rho-1}\, W_{k,\mu}(\beta x) M_{\lambda,\nu}\left(\frac{x^{-\delta/t}}{z}\right) dx$$

$$=(2\pi)^{\frac{1}{2}(1-t)+\frac{1}{2}-\frac{1}{2}\delta}\, t^{\lambda-\nu}\, \delta^{k+\rho-\frac{\delta}{t}(\frac{1}{2}+\nu)-\frac{1}{2}}\, \alpha^{-\rho+\frac{\delta}{t}(\frac{1}{2}+\nu)-k}\, \beta^k\, z^{-\frac{1}{2}-\nu}$$

$$\times \frac{\Gamma(1+2\nu)}{\Gamma(\frac{1}{2}-k+\mu)\,\Gamma(\frac{1}{2}-k-\mu)\,\Gamma(\frac{1}{2}+\lambda+\nu)} \sum_{r=0}^{\infty} \frac{\Gamma(\frac{1}{2}-k+\mu+r)\,\Gamma(\frac{1}{2}-k-\mu+r)}{\lfloor r} \left(\frac{\beta-\alpha}{\beta\delta}\right)^r$$

$$\times G^{t,\, t+2\delta}_{2t+2\delta,\, t+\delta}\left(\frac{z^t t^t \delta^\delta}{\alpha^\delta}\,\middle|\, \begin{matrix}\Delta(\delta,\frac{1}{2}+\mu-\rho+\frac{\delta}{t}(\frac{1}{2}+\nu)),\ \Delta(\delta,\frac{1}{2}-\mu-\rho+\frac{\delta}{t}(\frac{1}{2}+\nu)),\ \Delta(t,1),\ \Delta(t,1+2\nu)\\ \Delta(t,\frac{1}{2}+\lambda+\nu),\ \Delta(\delta,k-\rho+\frac{\delta}{t}(\frac{1}{2}+\nu)-r)\end{matrix}\right)$$

where δ and t are positive integers,

$$|\arg z| < \frac{\pi}{2},\ Re(\alpha) > 0,\ Re(\beta) > \tfrac{1}{2}Re(\alpha),\ Re(\rho+\tfrac{1}{2}\pm\mu+\tfrac{\delta}{t}\lambda) > 0.$$

3·3. Particular cases of the intergal (2·2).

(*i*) In (2·2), replacing α,β,λ,μ by $\frac{1}{2}+\nu$, $\frac{1}{2}-\nu$, $\frac{1}{2}+\mu$, $\frac{1}{2}-\mu$ respectively and using [3, p. 351, (14)], we have

$$(3\cdot 10)\qquad \int_0^\infty e^{\frac{1}{2}x(\zeta-\sigma)}\, x^\rho\, K_\nu\left(\frac{\sigma x}{2}\right) K_\mu\left(\frac{\zeta x}{2}\right) G^{m,n}_{p,q}\left(zx^{\delta/t}\,\middle|\, \begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right) dx$$

$$=(2\pi)^{(1-t)(m+n-\frac{1}{2}p-\frac{1}{2}q)+\frac{1}{2}-\frac{1}{2}\delta}\, t^{\Sigma b_q-\Sigma a_p+\frac{1}{2}p-\frac{1}{2}q+1}\, \delta^{\rho-\frac{1}{2}}\, \sigma^{-\rho-\frac{1}{2}}$$

$$\times \frac{\cos\nu\pi\cos\mu\pi}{\pi\,(\zeta)^{\frac{1}{2}}}\, \Gamma(\tfrac{1}{2}-\nu) \sum_{r=0}^{\infty} \frac{\Gamma(\frac{1}{2}+\nu+r)}{\lfloor r}\, \delta^{-r} \sum_{u=0}^{\infty} \frac{\Gamma(\frac{1}{2}+\mu+u)\,\Gamma(\frac{1}{2}-\mu+u+r)}{\lfloor u} \left(\frac{\zeta-\sigma}{\zeta\sigma}\right)^u$$

$$\times G^{tm,\, tn+3\delta}_{tp+3\delta,\, tq+2\delta}\left(\frac{z^t t^{t(p-q)}\delta^\delta}{\sigma^\delta}\,\middle|\, \begin{matrix}\Delta(\delta,-\nu-\mu-\rho),\Delta(\delta,\nu+\mu-\rho),\Delta(\delta,-\nu+\mu-\rho-r),\Delta(t,a_1),\ldots,\Delta(t,a_p)\\ \Delta(t,b_1),\ \Delta(t,b_q),\ \Delta(\delta,-\frac{1}{2}+\mu-\rho-r),\ \Delta(\delta,-\frac{1}{2}-\nu-\rho-r-u)\end{matrix}\right)$$

where δ and t are positive integers,

$$2(m+n) > p+q,\ |\arg z| < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi,\ Re(\sigma) > 0,\ Re(\zeta) > \tfrac{1}{2}Re(\sigma),$$

$$Re(\rho+1\pm\nu\pm\mu+\tfrac{\delta}{t}b_j) > 0,\ j=1,2,\ldots,m.$$

When $\sigma=\zeta=1$, $\delta=t=1$, the series (3·10) can be summed.

Expressing the G-function on the right hand side as Mellin-Barnes type integral [1, p. 207, (1)], and interchanging the order of integration and summation, we have

$$\frac{\cos\nu\pi\cos\mu\pi}{\pi}\Gamma(\tfrac{1}{2}+\nu)\Gamma(\tfrac{1}{2}-\nu)\Gamma(\tfrac{1}{2}+\mu)\Gamma(\tfrac{1}{2}-\mu)\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}$$

$$\times \frac{\Gamma(1+\nu+\mu+\rho+s)\Gamma(1-\nu-\mu+\rho+s)\Gamma(1+\nu-\mu+\rho+s)}{\Gamma(\tfrac{3}{2}-\mu+\rho+s)\Gamma(\tfrac{3}{2}+\nu+\rho+s)}$$

$$\times {}_3F_2\left(\begin{matrix}1+\nu-\mu+\rho+s,\tfrac{1}{2}+\nu,\tfrac{1}{2}-\mu\\ \tfrac{3}{2}-\mu+\rho+s\ \tfrac{3}{2}+\nu+\rho+s\end{matrix};1\right)z^s\,ds$$

Applying Dixon's theorem [3, p. 362, (5)] to the Hypergeometric function, replacing s by $2s$, using multiplication formula for gamma-function, then using [1, p. 207 (1)], we obtain the result

$$(3{\cdot}11)\qquad \int_0^\infty x^\rho K_\nu\left(\frac{x}{2}\right)K_\mu\left(\frac{x}{2}\right)G^{m,n}_{p,q}\left(zx\,\Big|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)dx$$

$$=(2\pi)^{\frac{1}{2}p+\frac{1}{2}q-m-n+\frac{1}{2}}\ 2^{\Sigma b_q-\Sigma a_p+\frac{1}{2}p-\frac{1}{2}q+\rho-\frac{1}{2}}$$

$$\times\ G^{2m,\,2n+4}_{2p+4,\,2q+2}$$

$$\left(z^2 2^{2(p-q+1)}\,\Bigg|\,\begin{matrix}\frac{1-\mu+\nu-\rho}{2},\ \frac{1-\mu-\nu-\rho}{2},\ \frac{1+\mu+\nu-\rho}{2},\ \frac{1+\mu-\nu-\rho}{2},\ \Delta(2,a_1),\ldots,\Delta(2,a_p)\\ \Delta(2,b_1),\ \ldots\ldots\ldots,\Delta(2,b_q),\Delta(2,-\rho)\end{matrix}\right)$$

where $2(m+n)>p+q$, $|\arg z|<(m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$,

$Re\ (\rho+1\pm\nu\pm\mu+b_j)>0,\ j=1,2,\ldots,m.$

However, as the product of two Bessels functions can be replaced by a G-function, the integral (3·11) is a special case of [7, p. 401, (8)].

(*ii*) In (2·2) replacing α,β,λ,μ by $\frac{1}{2}-k+\mu$, $\frac{1}{2}-k-\mu$, $\frac{1}{2}-\lambda+\nu$, $\frac{1}{2}-\lambda-\nu$ respectively and using [3, p. 351, (15)], we have

$$(3{\cdot}12)\qquad \int_0^\infty e^{\frac{1}{2}x(\zeta-\sigma)}\,x^{\rho-k-\lambda-1}\,W_{k,\mu}(\sigma x)\ W_{\lambda,\nu}(\zeta x)\ G^{m,n}_{p,q}\left(zx^{\delta/t}\,\Big|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)dx$$

$$=(2\pi)^{(1-t)(m+n-\frac{1}{2}p-\frac{1}{2}q)+\frac{1}{2}-\frac{1}{2}\delta}\ t^{\Sigma b_q-\Sigma a_p+\frac{1}{2}p-\frac{1}{2}q+1}\ \delta^{\rho-\frac{1}{2}}\sigma^{-\rho+k}\zeta^{\lambda}$$

$$\times\frac{1}{\Gamma(\frac{1}{2}-k+\mu)\Gamma(\frac{1}{2}-\lambda+\nu)\Gamma(\frac{1}{2}-\lambda-\nu)}\sum_{r=0}^\infty\frac{\Gamma(\frac{1}{2}-k+\mu+r)}{\lfloor r}\delta^{-r}$$

$$\times\sum_{u=0}^\infty\frac{\Gamma(\frac{1}{2}-\lambda+\nu+u)\Gamma(\frac{1}{2}-\lambda-\nu+u+r)}{\lfloor u}\left(\frac{\zeta-\sigma}{\zeta\sigma}\right)^u$$

$$\times\ G^{tm,\ tn+3\delta}_{tp+3\delta,\ tq+\delta}$$

$$\left(\frac{z^t t^{t(p-q)}\delta^\delta}{\sigma^\delta}\,\Bigg|\,\begin{matrix}\Delta(\delta,k+\lambda-\mu-\nu-\rho),\ \Delta(\delta,k+\lambda+\mu+\nu-\rho),\ \Delta(\delta,k+\lambda-\mu+\nu-\rho-r),\ \Delta(t,a_1),\ldots,\Delta(t,a_p)\\ \Delta(t,b_1),\ldots,\Delta(t,b_q),\ \Delta(\delta,-\frac{1}{2}+2k+\lambda+\nu-\rho-r),\ \Delta(\delta,-\frac{1}{2}+2\lambda+k-\mu-\rho-r-u)\end{matrix}\right)$$

A-2

where δ and t are positive integers,

$2(m+n)>p+q$, $|\arg z|<(m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$, $Re(\sigma)>0$, $Re(\zeta)>\frac{1}{2}Re(\sigma)$,

$Re(\rho+1-k-\lambda\pm\mu\pm\nu+\frac{\delta}{t}b_j)>0, j=1,2,\ldots,m.$

In (3·12), putting $\sigma=\zeta=1$, $\delta=t=1$, $\lambda=-k$, $\nu=\mu$ and applying the same procedure, which is used to derive (3·11), except using Whipple's theorem [3, p. 364, (2)] instead of Dixon's theorem, we obtained the following result.

$$(3\cdot13)\qquad \int_0^\infty x^{\rho-1} W_{k,\mu}(x)\; W_{-k,\mu}(x)\; G^{m,n}_{p,q}\left(zx\,\Big|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)dx$$

$$=(2\pi)^{\frac{1}{2}p-\frac{1}{2}q-m-n-\frac{1}{2}}\,2^{\Sigma b_q-\Sigma a_p+\frac{1}{2}p-\frac{1}{2}q+\rho+\frac{1}{2}}$$

$$\times\, G^{2m,\,2n+4}_{2p+4,\,2q+2}\left(z^2 2^{2(p-q+1)}\,\Bigg|\,\begin{matrix}\frac{1-2\mu-\rho}{2},\ \frac{1+2\mu-\rho}{2},\ \Delta(2,-\rho),\ \Delta(2,a_1),\ldots,\Delta(2,a_p)\\ \Delta(2,b_1),\ldots,\Delta(2,b_q),\ k-\frac{\rho}{2},\ -k-\frac{\rho}{2}\end{matrix}\right)$$

where $2(m+n)>p+q$, $|\arg z|<(m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$,

$Re(\rho+1\pm2\mu+b_j)>0,\ j=1,2,\ldots,m.$

Again, as the product of two Whittaker functions involved in the above integral can be expressed as a G-function, the integral (3·13) is a special case of [7, p. 401, (8)].

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REFERENCES

1. Erdelyi, A. Higher Transcendental functions Vol I (1953)
2. Erdelyi, A. Tables of Integral Transforms Vol II (1954)
3. Macrobert, T. M. Functions of a Complex Variable (1962)
4. Macrobert, T. M Some Integrals involving E-function. *Pro. Glasg. Math. Ass.*, **1**, 190-191 (1953)
5. Saxena, R. K. Some theorems in operational calculus and infinite integrals involving Bessel function and G functions. *Proc. Nat. Inst. Sci. India*, **27 A** : 38-61 (1961)
6. Saxena, R. K A study of the generalized Stielties transform. *Proc. Nat. Inst. Sci, India* **25 A**, 340-355 (1959)
7. Saxena, R. K. Some theorems of generalized Laplace transform—I. *Proc. Nat. Acad. Sci., India*, **26 A** 400-413 (1960)

APPLICATION OF NEW METHOD TO THE LAMINAR BOUNDARY LAYER EQUATIONS

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ABSTRACT

The new method as introduced by Kennedy [4] has been extended to the laminar flow through the entrance region of a channel bounded by two parallel flat walls. In § 1 the solution is obtained in terms of Airy Integrals and then expanded in series The series solution is compared with the Blasius solution and it has been found that the drag coefficient for small x as calculated by the present solution is about six percent less than the well known result. In § 2, the method is extended to the laminar jet mixing region of two uniform streams.

INTRODUCTION

The problem considered is that of obtaining approximations to the certain laminar-boundary layer equations which are reduced to the Blasius equations. Such equations are solved numerically and tables are available but the best analytic solution is the divergent series solution, obtained in 1908 [1]. During the last fifty-five years several authors have done work in this field but no one has much improved the divergent series solution of 1908. Notable work in this direction are by Weyl [2] who described a solution of the differential equations of some boundary layer problems ; Karman [3] classified laminar boundary layers with the solved problems in fluid mechanics on the basis of another solution ; Kennedy [4] has added one term in the Blasius equation and obtained the series solution in terms of variable η which connects x and y coordinates.

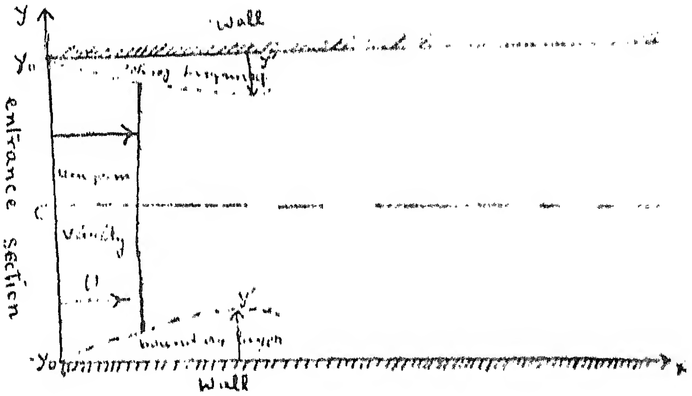
In the present note, the work of reference 4 has been extended to the work of Schlichting [5] for the laminar flow in the entrance region of a channel bounded by two parallel flat walls. It is also extended to the laminar jet mixing region of two parallel streams. The method applicable to the non-Newtonian fluids flow is under study by the author.

§ 1. THE BOUNDARY LAYER IN THE ENTRANCE SECTION OF A CHANNEL BOUNDED BY TWO PARALLEL FLAT WALLS

Consider the flow in the entrance section of a channel bounded by two parallel flat walls. All the entrance section, $x = O$, $u = V =$ constant and $v=O$. As x increases, due to the viscous effect, the flow is retarded in the region near the wall but is only affected a little in the central portion of the channel. For small values of x, the flow near the wall behaves like the boundary layer over a flat plate but with a longtitudinal pressure gradient. The solution is obtained [5] by the series expansion. As x increases to very large value, the velocity distribution will become parabolic. The solution for the entrance region when x is small is obtained as follows [5].

The equation of continuity gives

$$\int_0^{y_0} u \, dy' = U y_0 \qquad (1 \cdot 1)$$



where $y' = y_0 - y$, y_0 = half the width of the channel.

Also, we have

$$\int_0^{y_0} (u_\infty - u) \, dy' = u_\infty \, \delta^* \qquad (1 \cdot 2)$$

where δ^* is displacement thickness.

Combining 1·1 and 1·2, we have

$$u\infty(x) = U \frac{y_0}{y_0 - \delta^*} = U \left[1 + \frac{\delta^*}{y_0} + \left(\frac{\delta^*}{y_0} \right)^2 + \ldots \right] \qquad (1 \cdot 3)$$

For the flat plate, the displacement thickness is known.

Thus

$$\frac{\delta^*}{y_0} = 1 \cdot 73 \sqrt{\frac{\nu x}{y_0^2 U}} = 1 \cdot 73 \, \varepsilon = k_1 \, \varepsilon \text{ (say)} \qquad (1 \cdot 4)$$

Thus equation 1·3, gives

$$u\infty(x) = U[1 + k_1 \varepsilon + k_2 \varepsilon^2 + k_3 \varepsilon^3 + \ldots] \qquad (1 \cdot 5)$$

where $k_1 = 1 \cdot 73$ and k_2, k_3, . . . are known from the series solution.

The boundary layer equations for a flow pass a flat plate at zero incidence are

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \qquad (1 \cdot 6)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1 \cdot 7)$$

The following non-dimensional variables are introduced

$$\eta = y' \sqrt{\frac{U}{\nu x}} \qquad (1 \cdot 8)$$

$$v(x, y') = U y_0 [\varepsilon F_0(\eta) + \varepsilon^2 F_1(\eta) + \varepsilon^3 F_2(\eta) + \ldots] \qquad (1 \cdot 9)$$

$$u(x, y') = U[F_0'(\eta) + \varepsilon F_1'(\eta) + \varepsilon^2 F_2'(\eta) + \ldots] \qquad (1 \cdot 10)$$

Now the equations 1·5, 1·8, 1·9, 1·10 are substituted into 1·6 and the co-efficient of ϵ are collected. Thus the following equations are obtained.

$$F_0 F_0'' + 2F_0''' = 0 \tag{1·11}$$

with the boundary condition

$$\left.\begin{array}{r} F_0 = F_0' = 0 \text{ at } \eta = 0 \\ F_0' = 1 \text{ at } \eta = \infty \end{array}\right\} \tag{1·12}$$

and

$$2F_1''' + F_0 F_1'' - F_0' F_1 + 2F_0'' F_1 = -k_1 \tag{1·13}$$

with the boundary conditions

$$\left.\begin{array}{r} F_1 = F_1' = 0 \text{ at } \eta = 0 \\ F_1' = k_1 \text{ at } \eta = \infty \end{array}\right\} \tag{1·14}$$

and so on.

The purpose of the present note is to find F_0 in terms of η for the equation 1·11 with the boundary conditions of the set 1·12. The series solution for 1·11 is known [1].

Equation 1·11 can be reduced to the equation of Kennedy (4, Eq. 1) (for convenience) by replacing $F_0(\eta)$ as some another function of $f_0(\eta)$

Thus 1·11 becomes

$$f_0''' + 2 f_0 f_0' = 0 \tag{1·15}$$

with the boundary conditions

$$\left.\begin{array}{r} f_0 = f_0' = 0 \text{ at } \eta = 0, \\ f_0' = 1 \text{ at } \eta = \infty. \end{array}\right\} \tag{1·16}$$

Adding $2f_0'^2$ on both sides of 1·15, we see that the left hand side becomes an exact differential. The additional term on the right hand side is given its value at any given point (origin) and is considered constant. Thus equation 1·15 is written as (ref. 4, eq. 2).

$$f_0''' + 2(f_0 f_0'' + f_0'^2) = 2f_0'^2 = 0 \tag{1·17}$$

Integrating 1·17 twice we get Riccati equation

$$f_0' + f_0^2 = \lambda\eta \tag{1·18}$$

where $\lambda = f_0''(0)$ and the first and second boundary conditions of 1·16 are satisfied. To solve 1·18 put

$$f_0 = \frac{u'}{u}$$

Thus we have

$$u'' - \lambda\eta\, u = 0 \tag{1·19}$$

whose solution in terms of Airy Integrals is

$$u = C_1\, A_i(\xi) + C_2\, B_i(\xi) \tag{1·20}$$

where $\xi = \lambda^{1/3}\eta$. Thus

$$f_0(\eta) = \frac{\sqrt{3}\, A_i'(\xi) + B_i'(\xi)}{\sqrt{3}\, A_i(\xi) + B_i(\xi)} \tag{1·21}$$

where a dash stands for the differentiation with respect to η. Thus from 1·15 and 1·21, we have

$$\frac{f_0'''}{f_0''} = -2f_0 = -2\left[\frac{\sqrt{3}\, A_i'(\xi) + B_i'(\xi)}{\sqrt{3}\, A_i(\xi) + B_i(\xi)}\right] \tag{1·22}$$

which on integrating twice gives

$$f_0' = \int_0^\eta f_0''\, d\eta = -\frac{A_i(0)}{A_i'(0)}\, \lambda^{2/3}\left[1 - \frac{2\sqrt{3}\, A_i(\xi)}{\sqrt{3}\, A_i(\xi) + B_i(\xi)}\right] \tag{1·23}$$

where the relation

$$A_i(x)\, B_i'(x) - A_i'(x)\, B_i(x) = \frac{1}{\pi} \tag{1·24}$$

has been used.

Thus
$$f_0'(\infty) = 1 = -\frac{A_i(0)}{A_i'(0)}\, \lambda^{2/3}$$

Since the limit $\lim_{x \to \infty} \frac{A_i(x)}{B_i(x)} = 0$

and so

$$\lambda = \left[-\frac{A_i'(0)}{A_i(0)}\right] = ·62244$$

Finally we have

$$f_0' = \left[1 - \frac{2\sqrt{3}A_i(\alpha\eta)}{\sqrt{3}\, A_i(\alpha\eta) + B_i(\alpha\eta)}\right] \tag{1·25}$$

where

$$\alpha = \left[-\frac{A_i'(0)}{A_i(0)}\right]^{\frac{1}{2}}$$

Equation 1·25 has been expanded and the following series solution is obtained

$$f_0'(\eta) = \lambda\eta - \frac{2}{4!}\, \lambda^2\, \eta^4 + \frac{50}{7!}\, \lambda^3\, \eta^7 - \frac{4780}{10!}\, \lambda^4\, \eta^{10} + \ldots \tag{1·26}$$

The Blasius series solution gives

$$f_0'(\eta) = \mu\, \eta - \frac{2}{4!}\, \mu^2\, \eta^4 + \frac{44}{7!}\, \mu^3\, \eta^7 - \frac{2900}{10!}\, \mu^4\, \eta^{10} + \ldots \tag{1·27}$$

where $\mu = ·66412$ as calculated by Howarth [7].

Thus we see that the work of Kennedy [4] is capable to the present problem.

From 1·26 we see that

$$f_0''(0) = \alpha = \cdot 311$$

Skin Friction

The skin friction can be easily determined from the preceding data. The skin friction is

$$D = b \int_{x=0}^{l} \tau_0(x)\, dx \qquad (1\cdot 28)$$

where b is the width and l the length of the plate.

The local shearing stress at the walls is given by

$$\tau_0(x) = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu U \sqrt{\frac{U}{\nu x}}\, f_0''(0) = \alpha\, \mu\, U \sqrt{\frac{U}{\nu x}} \qquad (1\cdot 29)$$

Hence the dimensionless shearing stress becomes

$$\frac{\tau_0(x)}{\rho U^2} = \cdot 311 \sqrt{\frac{\nu}{Ux}} = \frac{\cdot 311}{\sqrt{R_x}} \qquad (1\cdot 30)$$

Consequently from 1·28, the skin friction for one wall becomes

$$D = \alpha \mu\, b\, U \sqrt{\frac{U}{\nu}} \int_{x=0}^{l} \frac{dx}{\sqrt{x}} = 2\, \alpha\, b\, U \sqrt{\mu \rho l U} \qquad (1\cdot 31)$$

Thus the dimensionless drag coefficient by the definition is

$$C_f = \frac{D}{\frac{1}{2} \rho A U^2},$$

where $A = bl$.

Thus

$$C_f = \frac{1\cdot 244}{\sqrt{R_l}} \qquad (1\cdot 32)$$

where $R_l = \frac{Ul}{\nu}$ denotes the Reynolds number formed with the length of the plate and the free streem velocity. This law of friction on a plate, first deduced by H. Blasius, is valid only in the region of laminar flow, *i.e.* for

$$R_l = \frac{Ul}{\nu} < 5 \times 10^5 \text{ to } 5 \times 10^6$$

The expression for C_f as calculated from 1·27 is well known expression

$$C_f = \frac{1\cdot 328}{\sqrt{R_l}} \qquad (1\cdot 33)$$

Thus the value of the drag coefficient as calculated from the use of Airy Integral to the Boundary Layer Equation is about 6 percent less.

§ 2. JET MIXING REGION OF LAMINAR FLOW OF TWO UNIFORM STREAM

In this case equations 1·1 and 1·2 hold with the boundary conditions

$$u(x, +\infty) = U_1,\ u(x, -\infty) = U_2 \tag{2·1}$$

Introducing

$$\nu\ (x, y) = U^{b_1} \nu^{b_2} x^{b_3} g(\eta),\ \eta = \frac{U^{b_4}\ y}{\nu^{b_5}\ x^{b_6}} \tag{2·2}$$

into 1·1, we get

$$b_1 = b_2 = \ldots = b_6 = \tfrac{1}{2},$$

so that the boundary conditions are satisfied.

The equation of motion in terms of $g(\eta)$ gives

$$g''' + \tfrac{1}{2}\, g\, g' = 0 \tag{2·3}$$

with the reduced boundary conditions

$$g'(\infty) = 1 + \lambda,\ g'(-\infty) = 1 - \lambda$$

where

$$\lambda = \frac{U_1 - U_2}{U_1 + U_2} \tag{2·4}$$

Equation 2·3 has been solved by Göertler [8] in series by putting

$$g(\eta) = 2 \sum_{m=0}^{\infty} \lambda^m g_m(\xi) \tag{2·5}$$

with

$$g_0 = \frac{\eta}{2} = \xi.$$

But the equation 2·3 is of the form 1·11 and thus the solution is obtainable with the use of Airy Integrals by introducing certain variables so that the boundary conditions are suitably changed.

The method extendable to the power law fluids flow and laminar free convection is under study of the author.

REFERENCES

1. Blasius. H. The Boundary Layers in Fluids with Little Friction. NACA TM 1256 (1950). *Translated from ZAMP. Band 56, Heft, 1 :* (1908).
2. Weyl, H. On the differential equations of the Simplest Boundary Layer Problems. *Annls of Mathematics*, 43 , 381, (1942).
3. Theodore, V. Karman. Solved and Unsolved Problems of High Speed Aerodynamics. *Proc. of the Conference on High Speed Aeronautics, Polytechnic Institute of Brooklyn, Brooklyn, N. Y.*, (1955).
4. Kennedy, E. D. Application of a New Method of Approximation in the Solution of Ordinary Differential Eqs. to the Blasius Equations. *Journal of Applied Mechanics* (ASME), *March*, p. 113, (1964).
5. Schlichting, H. Laminare Kanaleinlaufströmung, ZAMM, 14 : 368, (1934).
6. Yang, K. T. Laminar Free Convection Wake Above a Heated Vertical Plate. *Journal of Applied Mechanics* (ASME) *March*, 131–138, (1964).
7. Goldstein, S. (Editor) Mordern Development in Fluid Dynamics. *Oxford University Press, Oxford, England.* (1938).
8. Göttler, H. Berechnung von Aufgaben der Freien Turbulenz auf Grund eines neuer Näherungsantzes, ZAMM, 22 : Nr. 5 Okt. 244–254, (1942).

TWO THEOREMS ON A GENERALIZATION OF LOMMEL AND MAITLAND TRANSFORMS

By

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ABSTRACT

In previous papers I have studied the properties of the generalized transform

$$f(x) = \int_0^\infty (xy)^{\frac{1}{2}} J_{\nu,\lambda}^{\mu}(xy)\, g(y)\, dy,$$

which reduces to Lommel transform (Hardy, 1925) for $\mu = 1$ and to the generalized Hankel transform (Agarwal, 1950) for $\lambda = 0$.

The present paper aims to establish two theorems connected with the above generalization and then to evaluate some integrals with the help of these theorems.

1. INTRODUCTION

In a recent paper* I have given a generalization to the Lommel transform (Hardy 1925) :

$$f(x) = \int_0^\infty (xy)^{\frac{1}{2}} F_\nu(xy)\, g(y)\, dy, \tag{1·1}$$

where

$$F_\nu(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (\frac{x}{2})^{\nu+2\lambda+2r}}{\Gamma(1+\lambda+r)\,\Gamma(1+\lambda+\nu+r)}$$

and to the Maitland transform (Agarwal 1950) :

$$f(x) = (\tfrac{1}{2})^\nu \int_0^\infty (xy)^{\nu+\frac{1}{2}}\, J_\nu^\mu\left(\frac{x^2 y}{4}\right) g(y)\, dy, \tag{1·2}$$

where

$$J_\nu^\mu(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!\,\Gamma(1+\nu+\mu r)}, \ (\mu>0),$$

is the Bessel-Maitland function ; by means of the integral equation

$$f(x) = \int_0^\infty (xy)^{\frac{1}{2}} J_{\nu,\lambda}^{\mu}(xy)\, g(y)\, dy$$

where

$$J_{\nu,\lambda}^{\mu}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (\frac{x}{2})^{\nu+2\lambda+2r}}{\Gamma(1+\lambda+r)\,\Gamma(1+\lambda+\nu+\mu r)}, \ (\mu>0).$$

*in the press

It is interesting to note that (1·3) reduces to (1·1) for $\mu=1$ and to (1·2) for $\lambda=0$. In some other papers I have studied some properties of (1·3).

In the present paper I have established two theorems connected with (1·3) and illustrated their application by means of some examples.

2. *Theorem* 1. If

$$f(x)=\int_0^\infty (xy)^{\frac{1}{2}}\, J_{\nu,\lambda}^{\mu}(xy)\, g(y)\, dy$$

and

$$\frac{2}{\mu}\, x^{\lambda(\mu-1)+\nu(\frac{\mu}{2}-1)+\frac{3}{4}\mu-2}\, g(x^{\mu/2}) \doteq x^{\frac{1}{2}\mu(\nu+1)-1}\, \psi(x^{\frac{1}{2}\mu}),$$

then

$$f(x)=\frac{\mu}{\sqrt{2}\,\Gamma(1+\lambda)}\left(\frac{x}{2}\right)^{\nu+2\lambda+\frac{1}{2}}\int_0^\infty y^{\rho-\frac{2}{\mu}(\nu+\lambda+1)}\, {}_1F_1\left(\begin{matrix}1\\1+\lambda\end{matrix};\frac{-x^2}{4y^2}\right)\psi(y)\, dy,$$

provided $\mu>0$, $R(\nu+\lambda)>-1$ and the integrals involved converge absolutely.

Proof : we have

$$f(x)=\int_0^\infty (xy)^{\frac{1}{2}}\, J_{\nu,\lambda}^{\mu}(xy)\, g(y)\, dy$$

$$=\int_0^\infty (xy)^{\frac{1}{2}}\, J_{\nu,\lambda}^{\mu}(xy)\, dy \times y^{\frac{2\lambda}{\mu}(1-\mu)+\frac{2\nu}{\mu}(1-\frac{1}{2}\mu)+\frac{3}{\mu}-\frac{3}{2}} \times \int_0^\infty t^{\rho}\exp\{-(yt)^{\frac{2}{\mu}}\}\,\psi(t)\, dt$$

$$= x^{\frac{1}{2}}\int_0^\infty t^{\rho}\,\psi(t)\, dt\int_0^\infty y^{\frac{2\lambda}{\mu}(1-\mu)+\frac{2\nu}{\mu}(1-\frac{1}{2}\mu)+\frac{3}{\mu}-1}$$

$$\times \exp\{-(yt)^{2/\mu}\}\, J_{\nu,\lambda}^{\mu}(xy)\, dy,$$

on changing the order of integration.

Now evaluating the last integral on the right, we get

$$f(x)=\frac{\mu x^{\nu+2\lambda+\frac{1}{2}}}{2^{\nu+2\lambda+1}\,\Gamma(1+\lambda)}\int_0^\infty t^{\rho-(2/\mu)(\nu+\lambda+1)}\, {}_1F_1\left(\begin{matrix}1\\1+\lambda\end{matrix};\frac{-x^2}{4t^2}\right)\psi(t)\, dt,$$

which proves the theorem.

It only remains to justify the change in the order of integration.

Now, we know (Wright 1935) that

(*i*) as $x\to\infty$

$$J_{\nu,\lambda}^{\mu}(x)\;\begin{cases} =O\left(x^{\nu+2\lambda-2k(\nu+2\lambda+\frac{1}{2})}\exp\left\{\left(\mu\dfrac{x^2}{4}\right)^{k}\dfrac{\cos\pi k}{\mu k}\right\}+\dfrac{x^{\nu+2\lambda-2}}{\Gamma(\lambda)\,\Gamma(\nu+\lambda-\mu+1)}\right), & (0<\mu\leqslant 1,\ k=1/1+\mu)\\[2ex] =O\left(x^{\nu+2\lambda-2k(\nu+2\lambda+\frac{1}{2})}\exp\left\{\left(\mu\dfrac{x^2}{4}\right)^{k}\dfrac{\cos\pi k}{\mu k}\right\}\right). & (\mu>1,\ k=1/1+\mu).\end{cases}$$

(*ii*) as $x \to 0$

$$J^{\mu}_{\nu,\lambda}(x) = O\,(x^{\nu+2\lambda}).$$

Therefore, both the t– and y– integrals converge absolutely, the double integral exists under the conditions stated above; hence by virtue of De La Vallee Poussin's theorem, the change in the order of integration is justified.

Note: For practical applications in the above theorem, we may write

$$x^{\lambda(\mu-1)+\nu(\mu/2-1)+(3/4)\mu-1}\, g(x^{\mu/2}) = \tfrac{1}{2}\mu \int_0^{\infty} y^{(\mu/2)(\rho+1)-1}\, e^{-xy}\, \psi\, y\,(^{\mu/2})\, dy.$$

Example: Let

$$\psi(x) = \frac{1}{(1+x^2)^{\alpha}} \text{ and } \mu = 1.$$

Then

$$f(x) = \frac{(x/_2)^{\nu+2\lambda+1/2}}{\sqrt{2}\,\Gamma(1+\lambda)} \int_0^{\infty} y^{\rho-2(\nu+\lambda+1)}\, {}_1F_1\left(\begin{matrix}1\\1+\lambda\end{matrix};\frac{-x^2}{4y^2}\right)\frac{dy}{(1+y^2)^{\alpha}}$$

Now evaluating the integral on the right, we get (Erdelyi (2), p. 236)

$$f(x) = \frac{x^{\nu+\lambda-\frac{1}{2}}}{2^{\nu+\lambda+1}\Gamma(\alpha)}\, G^{2\,2}_{2\,3}\left(\frac{x^2}{4}\,\middle|\,\begin{matrix}1-\nu-\alpha+\frac{1}{2}\rho-\frac{1}{2}\lambda,\ \frac{1}{2}+\frac{1}{2}\lambda\\ -\nu+\frac{1}{2}\rho-\frac{1}{2}\lambda,\ \frac{1}{2}+\frac{1}{2}\lambda,\ \frac{1}{2}-\frac{1}{2}\lambda\end{matrix}\right),$$

where $R(\nu+\alpha+\lambda+\frac{1}{2}) > \frac{1}{2}R(\rho) > R(\nu+\lambda)-\frac{1}{2}$.

Also, $g(x) = \frac{1}{2}\Gamma\left(\frac{\rho+1}{2}\right) x^{2\nu+\alpha-\frac{1}{2}\rho-2}\, e^{\frac{1}{2}x^2}\, W_{k,m}(x^2)$,

where $R(\rho) > -1$, $2k = \frac{1}{2} - \frac{1}{2}\rho - \alpha$ and $2m = \frac{1}{2} + \frac{1}{2}\rho - \alpha$,

by (Erdelyi (2), p. 234).

Then, from the above theorem, we get

$$\int_0^{\infty} y^{2\nu+\alpha-\frac{1}{2}\rho-3/2}\, s_{\nu+2\lambda-1;\nu}(xy)\, e^{\frac{1}{2}y^2}\, W_{k,m}(y^2)\, dy$$

$$= \frac{\Gamma(\lambda)\,\Gamma(\nu+\lambda)}{\Gamma(\alpha)\,\Gamma\left(\frac{1+\rho}{2}\right)}\, 2^{\lambda-2}\, x^{\nu+\lambda-1}\, G^{2\,2}_{2\,3}\left(\frac{x^2}{4}\,\middle|\,\begin{matrix}1-\nu-\alpha+\frac{1}{2}\rho-\frac{1}{2}\lambda,\ \frac{1}{2}+\lambda/2\\ -\nu+\frac{1}{2}\rho-\frac{1}{2}\lambda,\ \frac{1}{2}+\lambda/2,\ \frac{1}{2}-\lambda/2\end{matrix}\right),$$

where $\max(-1,\ 2R(\nu+\lambda)-1) < R(\rho) < 2R(\nu+\alpha+\lambda)+1,$

$R[2\lambda+3\nu+\alpha-\frac{1}{2}\rho \pm (\frac{1}{2}\rho-\alpha)] > 0,\ R(2\nu-\rho) < \min\,[R(2-\nu-2\lambda),\frac{1}{2}],$

$R(\nu+\lambda) > -1$ and $\neq 0$, $R(\lambda) \neq 0, -1, -2, \ldots,$

$2k = \frac{1}{2} - \frac{1}{2}\rho - \alpha$ and $2m = \frac{1}{2} + \frac{1}{2}\rho - \alpha.$

Then

$$x^{(\mu/2)\,(\nu+2\lambda+3/2)-\nu-\lambda-1}\; g(x^{\mu/2})$$

$$= \frac{2^{\lambda-2}}{\sqrt{\pi}}\, x^{-\nu-\lambda-1}\; G^{m,n+3}_{p+3,q}\left(\frac{4\alpha}{x}\,\middle|\,\begin{matrix} -\nu-\lambda,\ -\lambda/2,\ \frac{1}{2}-\lambda/2,\ a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right),$$

where $\quad R(b_j+\nu+\lambda) > -1,\ j=1, \ldots, m$, by Erdelyi (2) p. 419).

Therefore, our theorem gives

$$\int_0^\infty y^{\nu+2\lambda-1}\, J^{\frac{1}{2}}_{\nu,\lambda}\left(\frac{x}{y}\right) G^{m,n+3}_{p+3,q}\left(4\alpha y^4\,\middle|\,\begin{matrix} -\nu-\lambda,\ -\lambda/2,\ \frac{1}{2}-\lambda/2,\ a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) dy$$

$$= \frac{x^{\nu+2\lambda}}{\pi^{\frac{1}{2}}\, 2^{\nu+3\lambda+2}}\, G^{m+2,\,n+2}_{p+2,\,q+2}\left(\frac{\alpha x^4}{16}\,\middle|\,\begin{matrix} 0,\ \frac{1}{2},\ a_1, \ldots, a_p \\ 0,\ \frac{1}{2},\ b_1, \ldots, b_q \end{matrix}\right),$$

where

$$p+q < 2(m+n),\ |\arg\alpha| < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi,$$

$$\max(-R(\nu+\lambda),\ \tfrac{1}{4}-\tfrac{1}{2}R(\lambda),\ R(a_j)) < 1$$
$$1 \leq j \leq n$$

and $\quad \min R(b_j) > \frac{1}{4}\max(-1-R(\lambda),\ -1),\ j = 1, \ldots, m.$

Example 3. Let

$$\psi(x) = x^{\frac{1}{2}\rho}\, J_\rho(ax^{\frac{1}{2}}),\ (a > 0,\ R(\rho) > -1).$$

Then Erdelyi (2), p. 225)

$$x^{-(\nu/2+\lambda+\frac{1}{2})}\, f(2\sqrt{x})$$

$$= \frac{\mu}{2\sqrt{2}}\left[\left(\frac{2}{a}\right)^{-2\lambda+\rho} \frac{\Gamma(\rho-\lambda)}{\Gamma(1+\lambda)}\ {}_1F_2\left(\begin{matrix} 1 \\ 1+\lambda-\rho,\ 1+\lambda \end{matrix};\ \frac{a^2x}{4}\right)\right.$$

$$\left. -\ \pi\operatorname{cosec}\{(\rho-\lambda)\pi\}\, x^{-\lambda+\rho/2}\, I_\rho(ax^{\frac{1}{2}})\right]$$

where $\quad a > 0,\ R(\rho-\lambda) > -1$ and $R(\rho-2\lambda) < \frac{3}{2}$.

Also,

$$x^{(1/\mu)\,(\nu+\lambda+1)-\lambda-3/2}\, \phi\left(\frac{1}{\sqrt{x}}\right)$$

$$= \frac{1}{x}\left(\frac{a}{2x}\right)^\rho \exp\left(\frac{-a^2}{4x}\right),\ (R(\rho) > -1).$$

Then

$$x^{(\mu/2)(\nu+2\lambda+3/2)}\, g(x^{\mu/2}) = \frac{\frac{1}{2}(a/2)^\rho}{x^{\mu(\rho-\lambda)}}\; G^{\mu}_{1+\nu+\lambda+\mu(\rho-\lambda)}\left(\frac{a^2}{4},\ x^{-\mu}\right),$$

where $\quad 0 < \mu < 1,\ R(\nu+\lambda+\mu(\rho-\lambda)) > -1$

and

$$G^{\mu}_{\lambda}(x) = \mu \sum_{r=0}^{\infty} \frac{(-x)^r\, \Gamma(\mu r + \lambda)}{r!},$$

is the generalized parabolic cylinder function (Gupta, 1948).

Then, our theorem gives

$$\int_0^{\infty} y^{-(\nu+2\rho+1)}\, J^{\mu}_{\nu,\lambda}(xy)\; G^{\mu}_{1+\nu+\lambda+\mu(\rho-\lambda)}\left(\frac{a^2}{4y^2}\right) dy$$

$$= \frac{\mu\, x^{\nu+2\lambda}}{a^{\rho}\, 2^{\nu+2\lambda+1-\rho}} \left[(2/a)^{\rho-2\lambda} \frac{\Gamma(\rho-\lambda)}{\Gamma(1+\lambda)}\; {}_1F_2\left(\begin{matrix}1\\1+\lambda-\rho,\ 1+\lambda\end{matrix};\frac{a^2x^2}{16}\right) \right.$$

$$\left. - \pi \operatorname{cosec}\{(\rho-\lambda)\pi\}\,(x/2)^{\rho-2\lambda}\, I_{\rho}\left(\frac{ax}{2}\right)\right]$$

where $a > 0$, $0 < \mu < 1$, $R(\rho-\lambda) > -1$, $R(\rho-2\lambda) < \frac{3}{2}$ and $R(\nu+\lambda+1) > \max[\mu R(\rho-\lambda), 0]$.

Example 4. Let $\psi(x) = \cos ax$, $a > 0$, $\mu = \frac{1}{2}$ and $|R(\lambda)| < 1$.

Then (Erdelyi (2), p. 221)

$$x^{-(\nu/2+\lambda+\frac{1}{2})} f(2\sqrt{x})$$

$$= \frac{1}{4\sqrt{2}}\left[\tfrac{1}{2}\Gamma(1-\lambda)\, x^{-\lambda}\{e^{iax}\Gamma(\lambda, iax) + e^{-iax}\Gamma(\lambda, -iax)\}\right].$$

Also, $x^{(1/\mu)(\nu+\lambda+1)-\lambda+3/2}\, \phi\left(\frac{1}{\sqrt{x}}\right) = \frac{x}{x^2 + a^2}$

Then (Erdelyi (1), p. 137)

$$x^{(\mu/2)(\nu+2\lambda+3/2)-\nu-\lambda-1}\, g(x^{\mu/2})$$

$$= \tfrac{1}{4} a^{-\nu-\lambda/2-2}\, \Gamma(1+\nu+\lambda/2)\exp\left(\frac{x}{a^2}\right)\Gamma\left(-\nu-\lambda/2, \frac{x}{a^2}\right)$$

where $R(\lambda + 2\nu) > -2$.

Therefore, our theorem gives

$$\int_0^{\infty} y^{3\nu+2\lambda+3}\exp\left(\frac{y^4}{a^2}\right)\Gamma\left(-\nu-\frac{\lambda}{2}, \frac{y^4}{a^2}\right) J^{\frac{1}{2}}_{\nu,\lambda}(xy)\, dy$$

$$= \frac{a^{2\nu+\lambda+2}}{2^{\nu+2}}\frac{\Gamma(1-\lambda)}{\Gamma(1+\nu+\lambda/2)}\, x^{\nu}\left\{ e^{iax^2/4}\,\Gamma\left(\lambda, \frac{iax^2}{4}\right) + e^{-iax^2/4}\,\Gamma\left(\lambda, \frac{-iax^2}{4}\right)\right\}$$

where $a > 0$, $R(\nu+\lambda) > -\max[R(\nu+2), 1]$ and $|R(\lambda)| < 1$.

Example 5. Let $\psi(x) = \sin ax$, $(a > 0)$, $\mu = \frac{1}{2}$ and $|R(\lambda)| < 1$.

Then (Erdelyi (2), p. 219)

$$x^{-(\nu/2+\lambda+\frac{1}{2})} f(2\sqrt{x}) = \frac{i}{8\sqrt{2}}\Gamma(1-\lambda)\; x^{-\lambda}\left[e^{-iax}\Gamma(\lambda, -iax) - e^{iax}\Gamma(\lambda, iax)\right]$$

Also,

$$x^{(1/\mu)(\nu+\lambda+1)-\lambda-1}\,\phi\left(\frac{1}{x}\right) \text{ [illegible]}$$

Then (Erdelyi (2), p. [illegible])

$$\text{[illegible]}$$

$$= \frac{1}{4}\,\frac{\Gamma(\frac{3}{2}+\lambda/2+\nu)}{a^{2\nu+\lambda+2}}\,\exp\left(\text{[illegible]}\right)\text{ [illegible] },\ (R(2\nu+\lambda) > -3)$$

Therefore, our theorem gives

$$\int_0^\infty y^{2\nu+2\lambda+2}\exp\left(\frac{y^2}{a^2}\right)\text{ [illegible] }dy$$

$$= \frac{i}{4}\,\frac{\Gamma(1-\lambda)}{\Gamma(\frac{3}{2}+\lambda/2+\nu)}\,a^{2\nu+\lambda+2}\left(\text{[illegible]}\right)^{\nu}\left[\text{[illegible]}\left(\lambda, \frac{-1\,a^2}{4}\right)\right.$$

$$\left.\text{[illegible]}\left(\lambda, \frac{1\,a^2}{4}\right)\right],$$

where $a > 0$, $R(\nu+\lambda)$ [illegible] max [illegible] and [illegible].

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REFERENCES

1. Agarwal, R. P. *Annals de la Soc. Scient. Bruxelles*, 64 [illegible].
2. Erdelyi, A. Tables of Integral Transforms I [illegible]
3. Erdelyi, A. Tables of Integral Transforms II [illegible]
4. Hardy, G. H. *Proc. London Math. Soc.*, 1% [illegible]
5. Meijer, C. S. *Nederl. Akad. Wetensch. Proc.*, 1[illegible] [illegible].
6. Wright, E. M. *J. London Math. Soc.*, 10 [illegible]

RADIAL MOTION INSIDE A VISCOUS ROTATING MAGNETIC STAR

By

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ABSTRACT

Radial motion inside a viscous rotating magnetic star is investigated. Assuming uniform density for the stellar material it is concluded that superposition of a small amount of rotation does not materially alter the conclusions arrived by the former author for viscous non rotating magnetic stars.

INTRODUCTION

In 1958 Babcock[1] catalogued highly selected 338 stars which he had observed with his differential analyzer and found circular polarization due to the Zeeman Effect. The properties of cosmic rays[2] and the observed polarization in light of distant stars both seem to require the existence of a field of the order of 10^{-6} gauss throughout large parts of the galactic plane. It is also well known that general magnetic field of the order of one gauss exists at the surface of the sun and the earth. A much stronger field of the order of two thousand gauss is prevalent in the sun spots. Even small magnetic fields, inspite of their small magnitudes, produce considerable interaction between the field and the conducting ionized matter on account of the large linear dimensions over which they operate.

Alfvén and others[3] have proposed magnetic theories for sunspots, prominences etc. Magnetogravitational theories[4] for the formation of the spiral arms of the galaxies have been proposed. The former author has elsewhere[5] proposed a theory on the origin of the solar system by considering the instability of radial oscillations of a variable magnetic star. Wrubel[6] is of the opinion that stellar fields may be relics from the time the star condensed from interstellar matter.

On theoretical grounds it may be conjectured that stars may have stronger fields in their interior or atmosphere. A number of investigators have imposed magnetic field of arbitrary geometry upon incompressible static stars.[7] An axi-symmetric form of a magnetic field which is strictly compatible with spherical boundries has been found by Prendergast. Chandrasekhar and Fermi[8] have shown that a uniform magnetic field tends to produce an obtate spheroid by contracting the sphere along the direction of the field. Chandrasekhar has also proved that a spherical symmetry of the configuration is in general incompatible with the presence of fluid motions and the magnetic field. He has also mentioned an exception to this.

Recently the former author has elsewhere[9] considered radial motion inside a viscous non-rotating magnetic star for different laws of density. In the present analysis rotation has been considered which is commonly observed in stars.

Since radial motion in a fixed direction of θ and ϕ are being considered[10,11]

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hence

$$\left.\begin{aligned} q_r &= \frac{dr}{dt} = q \qquad \text{(say)} \\ q_\theta &= 0 \\ q_\phi &= r \sin\theta\, \omega. \end{aligned}\right\} \tag{1}$$

In the light of the above results the general hydrodynamical equation is

$$\frac{\partial q}{\partial t} + q\frac{\partial q}{\partial t} - \frac{q^2}{r} = F_r - \frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 q}{\partial r^2} + \frac{2}{r}\frac{\partial q}{\partial r} - \frac{2q}{r^2}\right] + r\omega^2 \sin^2\theta. \tag{2}$$

where the notations have their usual meaning. F_r is the component of the external force along the radius vector and which consists of the gravitational force and the magnetic force, which will be discussed under different heads. Electric force, however, will not be considered because of the infinite electrical conductivity assumed for the stellar material.

Gravitational force:

A rotating star can be more accurately represented by an oblate spheroid of small ellipticity ε. Let a, a and $a(1-\varepsilon)$ be the semi major and semi minor axes of the meridian section of an oblate spheroid.

The polar equation of the surface,[12] on neglecting the square of ellipticity, is

$$r = a(1 - \varepsilon\cos^2\theta) \tag{3}$$

the approximation being valid as according to Clairaut's theorem on rotating bodies.

$$\varepsilon\, \alpha\, \frac{\omega^2}{2\pi G\bar{\rho}}$$

where $\bar{\rho}$ is the mean density.

The components of the resultant attraction[13], at an internal point, are given by

$$\left.\begin{aligned} X &= -\tfrac{4}{3}\pi\rho\left(1 - \tfrac{2}{5}\varepsilon\right) f \\ Y &= -\tfrac{4}{3}\pi\rho\left(1 - \tfrac{2}{5}\varepsilon\right) g \\ Z &= -\tfrac{4}{3}\pi\rho\left(1 + \tfrac{4}{5}\varepsilon\right) h \end{aligned}\right\} \tag{4}$$

where ρ is the uniform density and (f, g, h) are the cartesian coordinates of the point.

Hence

$$\left.\begin{aligned} X &= -\tfrac{4}{3}\pi\rho\left(1 - \tfrac{2}{5}\varepsilon\right) r\sin\theta\cos\phi \\ Y &= -\tfrac{4}{3}\pi\rho\left(1 - \tfrac{2}{5}\varepsilon\right) r\sin\theta\sin\phi \\ Z &= -\tfrac{4}{3}\pi\rho\left(1 + \tfrac{4}{5}\varepsilon\right) r\cos\theta \end{aligned}\right\} \tag{5}$$

$$\begin{aligned} \text{Required Attraction} &= \tfrac{4}{3}\pi\rho\left[\left(1 - \tfrac{2}{5}\varepsilon\right)^2\{\sin^2\theta\cos^2\phi \right. \\ &\quad \left. + \sin^2\theta\sin^2\phi\} + \left(1 + \tfrac{4}{5}\varepsilon\right)^2\cos^2\theta\right]^{\frac{1}{2}} r \\ &= \tfrac{4}{3}\pi\rho\left[1 + \frac{2\varepsilon}{5}(3\cos^2\theta - 1)\right] r \end{aligned}$$

Magnetic force :

The magnetic field is assumed to be derived from a scalar potential of the form $\frac{S_n}{r^{n+1}}$ where S_n is the surface harmonic of degree indicated by the subscript and satisfies the Laplace's Equation, n in the above expression is a positive integer. Since we are considering uniform rotation about the Z-axis, the magnetic field will just rotate without suffering any distortion. Assuming that the magnetic field is constant along a line of force, but to vary from one line of force to another, it can be seen that the effect of the magnetic field is to increase the pressure by $\frac{\mu H^2}{8\pi}$. Thus the total pressure will be $p + \frac{\mu H^2}{8\pi}$, where μ is the magnetic permeability. Since radial motion in a fixed direction of θ and ϕ are being considered hence l is taken to be a constant. A simple possible form for the magnetic field[5] is

$$H = \frac{l}{r^{n+2}} \tag{7}$$

The relation between pressure and density for an adiabatic change is

$$p = k\rho^{\gamma} \tag{8}$$

where k is some constant, γ is the ratio of the specific heats (regarding the matter and enclosed radiation as one system) p and ρ are pressure and density respectively.

Let ν be the kinematic coefficient of viscosity and β the coefficient of viscosity[10], then

$$\nu = \frac{\beta}{\rho} \tag{9}$$

Now (2), and (6) to (9) give

$$\frac{\partial q}{\partial t} + q\frac{\partial q}{\partial r} - \frac{q^2}{r} = -\frac{4\pi\rho}{3}\left[1 + \tfrac{3}{5}\varepsilon\,(3\cos^2\theta - 1)\right] - \frac{1}{\rho}\frac{\partial}{\partial r}\left(k\rho^{\gamma} + \frac{\mu l^2}{8\pi r^{2n+4}}\right)$$
$$+ \frac{\beta}{\rho}\left(\frac{\partial^2 q}{\partial r^2} + \frac{2}{r}\frac{\partial q}{\partial r} - \frac{2q}{r}\right) + r\omega^2\sin^2\theta.$$

In steady state $\frac{\partial q}{\partial t} = 0$, and multiplying by $\frac{2}{r^2}$

$$\frac{d}{dt}\left(\frac{q^2}{r^2}\right) = -\frac{2}{r}\cdot\frac{4\pi\rho}{3}\left[1 + \tfrac{3}{5}\varepsilon\,(3\cos^2\delta - 1)\right] + \frac{(n+2)\mu l^2}{2\pi\rho}\cdot\frac{1}{r^{2n+7}}$$
$$+ \frac{2\omega^2\sin^2\theta}{r} + \frac{2\beta}{\rho r^4}\left(r^2\frac{d^2q}{dr^2} + 2r\frac{dq}{dr} - 2q\right) \tag{10}$$

The L. H. S. of (10) can be neglected for small motion. Multiplying (10) by $\frac{\rho r^4}{2\beta}$ and rearranging terms.

$$r^2 \frac{d^2q}{dr^2} + 2r\frac{dq}{dr} - 2q = \left(\frac{4\pi\rho^2}{3\beta}\right) r^3 - \frac{(n+2)\mu l^2}{2\pi\rho} \cdot \frac{\rho\, r^4}{2\beta\, r^{2n+7}}$$

$$+ \left[\frac{8\pi\rho^2\varepsilon}{15\beta}(3\cos^2\theta - 1) - \frac{\rho\omega^2}{\beta}\sin^2\theta\right] r^3 \tag{11}$$

The *C. F.* is $A e^{z} + B e^{2z}$, where A and B are arbitrary constants, and $e^{z} = r$.

$$\text{Now } P.\,I. = \left(\frac{4\pi\rho^2}{30\beta}\right) r^3 - \frac{\mu\, l^2\,(n+2)}{2.\,4\pi\beta(n+2)\,(2n+1)}\,\frac{1}{r^{2n+3}}$$

$$+ \left[\frac{8\pi\rho^2\varepsilon}{150\beta}(3\cos^2\theta - 1) - \frac{\rho\omega^2}{10\beta}\sin^2\theta\right] r^3$$

The solution of equation (11) is thus

$$q = -Ar + \frac{\beta}{r^2} + \left(\frac{2\pi\rho^2}{15\beta}\right) r^3 - \left[\frac{\mu\, l^2}{8\pi\,\beta(2n+1)}\right]\frac{1}{r^{2n+3}}$$

$$+ \left[\frac{4\pi\rho^2\varepsilon}{75\beta}(3\cos^2\theta - 1) - \frac{\rho\omega^2}{10\beta}\sin^2\theta\right] r^3 \tag{12}$$

Initially considering radial motion along the equatorial plane, *i.e.* $\theta = \pi/2$,

$$q = -Ar + \frac{B}{r^2} - \frac{a}{r^{2n+3}} + b\, r^3 + c\, r^3 \tag{13}$$

where

$$\left.\begin{aligned} a &= \frac{\mu l^2}{8\pi\beta\,(2n+1)} \\ b &= \frac{2\pi\,\rho^2}{15\beta} \\ c &= \frac{8\pi\,\rho^2\varepsilon}{75\beta} \end{aligned}\right\} \tag{14}$$

Applying the following conditions the value of A and B can be determined.

When $r = \xi$ then $q = 0$

and $r = \eta$ then $q = 0$

where ξ and η are the equatorial radius of a finite core (supposed small), and the equatorial radius of the star. It is assumed that velocity of the material along the radius vanishes at the core and at the surface of the star

$$-A\xi + \frac{\beta}{\xi^2} - \frac{a}{\xi^{2n+3}} + b\xi^3 + c\xi^3 = 0 \tag{15}$$

$$-A\eta+\frac{B}{\eta^2}-\frac{a}{\eta^{2n+3}}+b\eta^3+c\eta^3=0 \tag{16}$$

Solving (15) and (16) for A and B, and neglecting higher terms of ξ/η.

$$A=\frac{a}{\eta^3\,\xi^{2n+1}}+b\eta^2+c\eta^2 \tag{17}$$

$$B=\frac{a}{\xi^{2n+1}}+b\xi^3\eta^2+c\xi^3\eta^2 \tag{18}$$

To find the maxima and minima of q, on differentiating (13).

$$\frac{dq}{dr}=A-\frac{2B}{r^3}+\frac{a(2n+3)}{r^{2n+4}}+3b\,r^2+3c\,r^2 \tag{19}$$

For maxima and minima of q, $\dfrac{dq}{dr}=0$.

i.e. $3br^{2n+6}+3cr^{2n+6}-A\,r^{2n+4}-2B\,r^{2n+1}+(2n+3)\,a=0.$ (20)

The equation has two changes of sign and hence can have at the most two real positive roots.

Let $f(r)=3br^{2n+6}+3\,cr^{2n+6}-A\,r^{2n+4}-2B\,r^{2n+1}+(2n+3)\,a$

hence $f(\xi)=3(b+c)\,\xi^{2n+6}-A\,\xi^{2n+4}-2B\,\xi^{2n+1}+(2n+3)a.$

On putting values of A and B from (17) and (18), simplifying, and neglecting higher power of ξ/η.

$$f(\xi)=-3(b+c)\eta^2\,\xi^{2n+4}\left[1-\frac{(2n+1)a}{3(b+c)\,\eta^2\,\xi^{2n+4}}\right] \tag{19'}$$

and $f(\eta)=3(b+c)\,\eta^{2n+6}-A\,\eta^{2n+4}-2B\,\eta^{2n+1}+(2n+3)a.$

On putting values of A and B from (17) and (18), simplifying, and neglecting higher power of ξ/η.

$$f(\eta)=2(b+c)\,\eta^{2n+6}\left[1-\frac{3a}{2(b+c)\,\eta^5\xi^{2n+1}}\right] \tag{20'}$$

Now if

$$\frac{(2n+1)a}{3(b+c)\,\eta^2\,\xi^{2n+4}}<1$$

Substituting values of a, b and c from (14)

$$i.e.\ \eta>\frac{\rho\sqrt{5\,\mu}}{4\pi\rho\,\xi^{n+2}\,[1+\frac{g}{b}e]} \tag{21}$$

then from (19′) and (20′) it can be seen that $f(\xi)$ and $f(\eta)$ will be negative and positive respectively provided that

$$\frac{(2n+1)a}{3(b+c)\,\eta^2\,\xi^{2n+1}}>\frac{3a}{2(b+c)\,\eta^5\,\xi^{2n+1}}$$

or

$$\left(\frac{\eta}{\xi}\right)^3>\frac{9}{2(2n+1)} \tag{22}$$

The inequality (21) and (22) will clearly be satisfied if η is sufficiently large compared with ξ.

It can be easily seen that the form of (22) is nearly the same as when rotation is not considered.[9]

Therefore $f(r) = 0$ must have at least one or an odd number of real positive roots between ξ and η. But from (20) it is clear that $f(r) = 0$ cannot have more than two real positive roots. Hence there is one and only one real positive root of $f(r) = 0$ which lies between ξ and η. Let it be r_1, then r_1 must satisfy.

$$3(b+c)\, r_1^{2n+6} - A\, r_1^{2n+4} - 2B\, r_1^{2n+1} + (2n+3)\, a = 0$$

Now velocities at the points close to ξ and n will be determined, *i.e.*, at $r = \xi + \delta$ and $r = \eta - \delta$ where δ is very small. Making use of (13)

$$[q]_{r=\xi+\delta} = \frac{1}{(\xi+\delta)^{2n+3}} \left[(b+c)\,(\xi+\delta)^{2n+6} - A(\xi+\delta)^{2n+4} + B(\xi+\delta)^{2n+1} - a\right]$$

On expanding and neglecting δ^2/ξ^2, using (15), substituting value of A and B from (17) and (18), and simplifying

$$[q]_{r=\xi+\delta} = -\,3(b+c)\,\eta^2\,\delta \left[1 - \frac{(2n+1)\,a}{3(b+c)\,\eta^2\,\xi^{2n+4}}\right] \tag{23}$$

Again using (13).

$$[q]_{r=\eta-\delta} = \frac{1}{(\eta-\delta)^{2n+3}} \left[(b+c)\,(\eta-\delta)^{2n+6} - A\,(\eta-\delta)^{2n+4} + B(\eta-\delta)^{2n+1} - a\right]$$

On expanding and neglecting δ^2/ξ^2, using (16), substituting value of A and B from (17) and (18), and simplifying

$$[q]_{r=\eta-\delta} = -\,2(b+c)\,\eta^2\,\delta\left[1 - \frac{3a}{2(b+c)\,\eta^5\,\xi^{2n+1}}\right] \tag{24}$$

Thus $q_{r=\xi+\delta}$ and $q_{r=\eta-\delta}$ are both negative

Since there is only one point $r = r_1$ at which $dq/dr = 0$, and that the velocities close to ξ and η *viz.* at $r = \xi+\delta$ and $r = \eta - \delta$ are both negative hence $r = r_1$, must give algebraically minimum value of q.

Thus the minimum value of q

$$[q]_{\min} = -\frac{1}{r_1^{2n+3}} \left[-(b+c)r_1^{2n+6} + A\, r_1^{\,2n+4} - B\, r_1^{\,2n+1} + a\right] \tag{25}$$

which numerically will represent the maximum value of q.

It can be seen that if $c = 0$ (*i.e.* if the term due to rotation is neglected) then it reduces to the result obtained elsewhere by the former author.

The results can be interpreted as follows : For a slowly rotating viscous magnetic star, along the equatorial plane the velocity remains negative throughout. Therefore the fluid inside the star will be moving radially from the surface towards the centre. Also the velocity is taken to be zero on the surface of the star and also on the surface of the inner core, and it has been found that at $r = r_1$ the velocity though negative is numerically maximum, therefore the matter

will have a tendency to contract or perhaps get contracted. Due to redistribution of matter the motion will not remain steady, thus the star will no longer be of uniform density. A similar argument is applicable for motion along the polar plane, or in general along any direction.

It is interesting to note that the conclusion arrived at are quite similar to that when rotation is not considered.[9] Thus the superposition of a small amount of rotation does not materially alter the conclusions arrived previously[9] for non rotating viscous magnetic stars. Rapid rotation of the star will not be considered since according to Sen[14] fast rotating Cepheids do not in general exist.

It is well known from the investigations of Drs. Kopal, Chandrasekhar and Eddington that the Cepheid variables are much less centrally condensed than the main-sequence stars. In fact, Kopal has stated that δ-Cephei F-5 stars approach the limit of homogeneity. In the light of the above statement we are quite justified in assuming uniform density of the stellar material for our model.

In another paper is proposed to study the radial motion of a viscous rotating magnetic star for variable density[9].

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REFERENCES

1. Babcock, H. W. 'A Catalogue of Magnetic Stars', *Ap J.* (Suppl.), 3, 141, (1958).
2. Deutsch, A J. *Handbuch Der Physik*, 51, 689, (1958).
3. Cowling, T G. *Magnetohydrodynamics*, IInd Ed., (19.8).
4. Ôki, T., *et al. Supp. Prog. Theo. Phy.*, No. 31, (1965).
5. Banerji, A. C. and Srivastava, K. M. *Proc. Nat. Acad. Sci.*, Sec. A, **XXXIII**, Part I, 125–148, (1963).
6. Deutsch, A J. *Handbuch Der Physik*, 51, 714, (1958).
7. Deutsch, A. J. *Handbuch Der Physik*, 51, 717, (1958).
8. Chandrasekher, S. and Fermi, E. *Ap. J.*, 116, (1953).
9. Banerji A. C. and Gurtu, V. K. *Proc. Nat. Acad. Sci.*, Sec. A, **XXXIV**, Part II, 105–132, (1964).
10. Ramsey, A. S. 'A Treaties on Hydromechanics,' Part II, (1935).
11. Bhatnagar, P. L. *Bull. Cal. Math Soc*, **Vol. 38**, No. 2, June, (1946), 93–94.
12. Routh, E. J. 'A Treatise on Analytical Statics,' **Vol. II** : 156, (19C8).
13. Ramsey, A. S. 'An introduction to the theory of Newtonian Attraction,' 166–170, (1940).
14. Sen, H K. *Proc. Nat Acad. Sci*, **Vol. 13**, Part 3, May, (1943).
15. Sen, H. K. *Proc. Nat. Acad Sci.*, **Vol. XII**, Part 4, 236, Nov., (1942).

CONTOUR INTEGRALS ASSOCIATED WITH CERTAIN GENERALISED HYPERGEOMETRIC FUNCTIONS

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ABSTRACT

In the present paper we give contour integral representations of various hypergeometric functions of two variables and three variables, the contour of integration in every case being a Pochhammer's double loop usually denoted by $(1+, 0+, 1-, 0-)$. It is also illustrated how these integrals can be used in the integration of the systems of hypergeometric partial differential equations represented by them, as well as in the derivation of the various hypergeometric transformations and cases of reducibility of the functions involved.

1. INTRODUCTION

As long ago as 1880, Picard pointed out that Appell's F_1 can be represented by a single integral in the form

$$(1\cdot1)\quad F_1(\alpha, \beta, \beta' ; \gamma ; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)} \int_0^1 u^{\alpha-1}\,(1-u)^{\gamma-\alpha-1}\,(1-ux)^{-\beta}(1-uy)^{-\beta'}\,du,$$

provided that $Re(\alpha) > 0$ and $Re(\gamma-\alpha) > 0$.

Single integral representations of the remaining functions, *viz.*, F_2, F_3 and F_4, were subsequently given by Erdelyi, [2, pp. 231-232], and the formulae are

$$(1\cdot2)\quad F_2(\rho+\rho'-1, \beta, \beta' ; \gamma, \gamma' ; x, y) = \frac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$$
$$\times \int_C (-t)^{-\rho}\,(t-1)^{-\rho'}\;{}_2F_1\!\left(\rho, \beta ; \gamma ; \frac{x}{t}\right)\;{}_2F_1\!\left(\rho', \beta' ; \gamma' ; \frac{y}{1-t}\right) dt,$$

$$(1\cdot3)\quad F_3(\alpha, \alpha', \beta, \beta' ; \rho+\rho' ; x, y) = \frac{\Gamma(1-\rho)\,\Gamma(1-\rho')\,\Gamma(\rho+\rho')}{(2\pi i)^2}$$
$$\times \int_C (-t)^{\rho-1}\,(t-1)^{\rho'-1}\;{}_2F_1(\alpha, \beta ; \rho ; t x)\;{}_2F_1[\alpha', \beta' ; \rho' ; (1-t)y]\,dt,$$

$$(1\cdot4)\quad F_4(\alpha, \beta ; \gamma, \gamma' ; x, y) = \frac{\Gamma(\gamma)\,\Gamma(\gamma')\,\Gamma(2-\gamma-\gamma')}{(2\pi i)^2}$$
$$\times \int_C (-t)^{-\gamma}\,(t-1)^{-\gamma'}\;{}_2F_1\!\left[\alpha, \beta ; \gamma+\gamma'-1 ; \frac{x}{t}+\frac{y}{1-t}\right] dt.$$

The contour of integration in every case is a Pochhammer's double loop $1+, 0-, 1-, 0-)$ along which $|t| > |x|$ and $|(1-t)| > |y|$ in (1·2), and $\left|\frac{x}{t}+\frac{y}{1-t}\right| < 1$ in the case of (1·4).

In § 3 of the present paper we give a new type of integrals associated with Appell's four functions. Consequences of these results are the contour integrals associated with the Lauricella's set of hypergeometric functions of three variables which we discuss in §§ 4 and 5.

The formulae of present paper are shown to be useful not only in the systematic integration of the hypergeometric partial differential equations associated with them but are also applied, in the last section to obtain certain transformations and cases of reducibility of the hypergeometric functions of three variables.

In what follows we make a free use of the well-known formula [10, p. 256]

$$(1{\cdot}5) \qquad \int_{[0\,;\,1]} (-t)^{\alpha-1}\,(t-1)^{\beta-1}\,dt = \frac{(2\pi i)^2}{\Gamma(1-\alpha)\,\Gamma(1-\beta)\,\Gamma(\alpha+\beta)}\,;$$

where the contour of integration is a Pochhammer's double loop containing the origin within one loop and the point $t=1$ within the other (*cf.* [3], p. 378).

2 DEFINITIONS

Making use of the familiar abbreviation

$$(\lambda, m) = \frac{\Gamma(\lambda+m)}{\Gamma(\lambda)}\,,$$

the hypergeometric functions of three variables belonging to Lauricella's set are defined as follows:

$$(2{\cdot}1) \qquad F_E(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2\,; \gamma_1, \gamma_2, \gamma_3\,, x, y, z)$$
$$= \sum \frac{(\alpha_1, m+n+p)\,(\beta_1, m)\,(\beta_2, n+p)}{(1, m)\,(1, n)\,(1, p)\,(\gamma_1, m)\,(\gamma_2, n)\,(\gamma_3, p)}\, x^m\, y^n\, z^p.$$
$$[r + (s + t) = 1]\,;$$

$$(2{\cdot}2) \qquad F_F(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1\,; \gamma_1, \gamma_2, \gamma_2\,; x, y, z)$$
$$= \sum \frac{(\alpha_1, m+n+p)\,(\beta_1, m+p)\,(\beta_2, n)}{(1, m)\,(1, n)\,(1, p)\,(\gamma_1, m)\,(\gamma_2, n+p)}\, x^m\, y^n\, z^p.$$
$$[rs = (1-s)\,(1-t)\,]\,;$$

$$(2{\cdot}3) \qquad F_G(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3\,; \gamma_1, \gamma_2, \gamma_2\,; x, y, z)$$
$$= \sum \frac{(\alpha_1, m+n+p)\,(\beta_1, m)\,(\beta_2, n)\,(\beta_3, p)}{(1, m)\,(1, n)\,(1, p)\,(\gamma_1, m)\,(\gamma_2, n+p)}\, x^m\, y^n\, z^p,$$
$$[r + s = 1 = r + t]\,;$$

$$(2{\cdot}4) \qquad F_K(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1, \gamma_1, \gamma_2, \gamma_3 : x, y, z)$$
$$= \sum \frac{(\alpha_1, m)\,(\alpha_2, n+p)\,(\beta_1, m+p)\,(\beta_2, n)}{(1, m)\,(1, n)\,(1, p)\,(\gamma_1, m)\,(\gamma_2, n+p)}\, x^m\, y^n\, z^p,$$
$$[(1-r)\,(1-s) = t]\,;$$

$$(2{\cdot}5) \qquad F_M(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1\,; \gamma_1, \gamma_2, \gamma_2\,; x, y, z)$$
$$= \sum \frac{(\alpha_1, m)\,(\alpha_2, n+p)\,(\beta_1, m+p)\,(\beta_2, n)}{(1, m)\,(1, n)\,(1, p)\,(\gamma_1, m)\,(\gamma_2, n+p)}\, x^m\, y^n\, z^p$$
$$[r + t = 1 = s]\,:$$

(2·6) $$F_N(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_1 ; \gamma_1, \gamma_2, \gamma_2 ; x, y, z)$$
$$= \sum \frac{(\alpha_1, m)\,(\alpha_2, n)\,(\alpha_3, p)\,(\beta_1, m+p)\,(\beta_2, n)}{(1, m)\,(1, n)\,(1, p)\,(\gamma_1, m)\,(\gamma_2, n+p)}\, x^m y^n z^p$$
$$[(1-r)\, s + (1-s)\, t = 0] ;$$

(2·7) $$F_P(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_1, \beta_2 ; \gamma_1, \gamma_2, \gamma_2 ; x, y, z)$$
$$= \sum \frac{(\alpha_1, m+p)\,(\alpha_2, n)\,(\beta_1, m+n)\,(\beta_2, p)}{(1, m)\,(1, n)\,(1, p)\,(\gamma_1, m)\,(\gamma_2, n+p)}\, x^m y^n z^p,$$
$$[(st - s - t)^2 = 4rst] ;$$

(2·8) $$F_R(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_2, \beta_1 ; \gamma_1, \gamma_2, \gamma_2 ; x, y, z)$$
$$= \sum \frac{(\alpha_1, m+p)\,(\alpha_2, n)\,(\beta_1, m+p)\,(\beta_2, n)}{(1, m)\,(1, n)\,(1, p)\,(\gamma_1, m)\,(\gamma_2, n)\,(\gamma_3, p)}\, x^m y^n z^p,$$
$$[s(1 - \sqrt{r})^2 + (1-s)\, t = 0] ;$$

(2·9) $$F_S(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3 ; \gamma_1, \gamma_1, \gamma_1 ; x, y, z)$$
$$= \sum \frac{(\alpha_1, m)\,(\alpha_2, n+p)\,(\beta_1, m)\,(\beta_2, n)\,(\beta_3, p)}{(1, m)\,(1, n)\,(1, p)\,(\gamma_1, m+n+p)}\, x^m y^n z^p,$$
$$[r + s = rs,\ s = t] ;$$

(2·10) $$F_T(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1 ; \gamma_1, \gamma_1, \gamma_1 ; x, y, z)$$
$$= \sum \frac{(\alpha_1, m)\,(\alpha_2, n+p)\,(\beta_1, m+p)\,(\beta_2, n)}{(1, m)\,(1, n)\,(1, p)\,(\gamma_1, m+n+p)}\, x^m y^n z^p,$$
$$[r - rs + s = t],$$

(2·11) $$H_A(\alpha, \beta, \beta' ; \gamma, \gamma', x, y, z)$$
$$= \sum \frac{(\alpha, m+p)\,(\beta, m+n)\,(\beta', n+p)}{(1, m)\,(1, n)\,(1, p)\,(\gamma, m)(\gamma', n+p)}\, x^m y^n z^p,$$
$$[r + s + t = 1 + st] ;$$

(2·12) $$H_B(\alpha, \beta, \beta' ; \gamma_1, \gamma_2, \gamma_3 ; x, y, z)$$
$$= \sum \frac{(\alpha, m+p)\,(\beta, m+n)\,(\beta', n+p)}{(1, m)\,(1, n)\,(1, p)\,(\gamma_1, m)\,(\gamma_2, n)\,(\gamma_3, p)}\, x^m y^n z^p,$$
$$[r + s + t + 2\sqrt{rst} = 1] ;$$

(2·13) $$H_C(\alpha, \beta, \beta' ; \gamma ; x, y, z)$$
$$= \sum \frac{(\alpha, m+p)\,(\beta, m+n)\,(\beta', n+p)}{(1, m)\,(1, n)\,(1, p)\,(\gamma, m+n+p)}\, x^m y^n z^p,$$
$$[r = s = t = 1] ;$$

where 'Σ' stands for triple summation extending over all positive integral values of m, n and p from zero to infinity, and for convergence, $|x| < r$, $|y| < s$, and $|z| < t$

For definitions of Lauricella's triple series F_A, F_B, F_C and F_D see [1, p. 114]; and for a detailed discussion of the various properties of the Saran's functions $F_E, F_F, \ldots\ldots\ldots, F_T$ and Srivastava's H_A, H_B and H_C see [4], [5], [6] [7], [8] and [9].

3. INTEGRALS ASSOCIATED WITH APPELL'S DOUBLE FUNCTIONS

From definition of Gauss's series we have

$$ {}_2F_1(\rho+\rho'-1, \beta ; \gamma ; z) = \sum_{m=0}^{\infty} \frac{(\rho+\rho'-1, m)\,(\beta, m)}{(\gamma, m)\,(1, m)} z^m $$

$$ = \sum_{m=0}^{\infty} \frac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')\,(\beta, m)}{(2\pi i)^2\,(\gamma, m)\,(1, m)} z^m \frac{(2\pi i)^2\,(-1)^m}{\Gamma(\rho+m)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')} $$

Therefore, on making use of the formula (1·5) we find that

(3·1) $$ {}_2F_1(\rho+\rho'-1, \beta ; \gamma ; z) = \frac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2} \times \int (-t)^{-\rho}\,(t-1)^{-\rho'}\,{}_2F_1\left(\rho, \beta ; \gamma ; \frac{z}{t}\right) dt, $$

where $|t| > |z|$ along the contour so as to justify the change of order of integration and summation.

Similarly, we have

(3·2) $$ {}_2F_1(\rho+\rho'-1), \beta ; \gamma ; z) = \frac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2} \times \int (-t)^{-\rho}\,(t-1)^{-\rho'}\,{}_2F_1\left(\rho', \beta ; \gamma ; \frac{z}{1-t}\right) dt, $$

where $|1-t| > |z|$ along the contour;

(3·3) $$ {}_2F_1(\alpha, \beta ; \rho+\rho' ; z) = \frac{\Gamma(1-\rho)\,\Gamma(1-\rho')\,\Gamma(\rho+\rho')}{(2\pi i)^2} \times \int (-t)^{\rho-1}\,(t-1)^{\rho'-1}\,{}_2F_1(\alpha, \beta ; \rho ; zt)\, dt, $$

where $|zt| < 1$ along the contour;

and

(3·4) $$ {}_2F_1(\alpha, \beta ; \rho+\rho' ; z) = \frac{\Gamma(1-\rho)\,\Gamma(1-\rho')\,\Gamma(\rho+\rho')}{(2\pi i)^2} \times \int (-t)^{\rho-1}\,(t-1)^{\rho'-1}\,{}_2F_1[\alpha, \beta ; \rho' ; z(1-t)]\, dt, $$

where $|z(1-t)| < 1$ along the contour.

From the definition of F_1 we know that

$$ F_1(\alpha, \beta, \beta' ; \gamma ; x, y) = \sum_{m=0}^{\infty} \frac{(\alpha, m)\,(\beta, m)}{(1, m)\,(\gamma, m)} x^m\,{}_2F_1(\alpha+m, \beta' ; \gamma+m ; y). $$

Using (3·1) we get

$$F_1(\rho+\rho'-1, \beta, \beta' : \gamma ; x, y) = \sum_{m=0}^{\infty} \frac{(\rho+\rho'-1, m)(\beta, m)\Gamma(\rho+m)\Gamma(\rho')\Gamma(2-\rho-\rho'-m)}{(1, m)(\gamma, m)(2\pi i)^2} x^m$$

$$\times \int (-t)^{-\rho-m} (t-1)^{-\rho'} {}_2F_1\left(\rho+m, \beta' ; \gamma ; \frac{y}{t}\right) dt,$$

$$= \frac{\Gamma(\rho)\Gamma(\rho')\Gamma(2-\rho-\rho')}{(2\pi i)^2} \int (-t)^{-\rho} (t-1)^{-\rho'} \sum_{m=0}^{\infty} \frac{(\beta, m)(\rho, m)}{(1, m)(\gamma, m)} (x/t)^m$$

$$\times {}_2F_1\left(\rho+m, \beta' ; \gamma+m ; \frac{y}{t}\right) dt,$$

the change in the order of integration and summation being permissible when $|t| > \max. (|x|, |y|)$ along the contour, and therefore

(3·5) $$F_1(\rho+\rho'-1, \beta, \beta' ; \gamma ; x, y) = \frac{\Gamma(\rho)\Gamma(\rho')\Gamma(2-\rho-\rho')}{(2\pi i)^2}$$

$$\times \int (-t)^{-\rho} (t-1)^{-\rho'} F_1\left(\rho, \beta, \beta' ; \gamma ; \frac{x}{t}, \frac{y}{t}\right) dt,$$

provided $|x/t| < 1$, $|y/t| < 1$ along the contour.

Similar consequences of the formulae (3·1), (3·2), (3·3) and (3·4) are the following contour integral representations of Appell's functions:

(3·6) $$F_1(\rho+\rho'-1, \beta, \beta' ; \gamma ; x, y) = \frac{\Gamma(\rho)\Gamma(\rho')\Gamma(2-\rho-\rho')}{(2\pi i)^2}$$

$$\times \int (-t)^{-\rho} (t-1)^{-\rho'} F_3\left(\rho, \rho', \beta, \beta' ; \gamma ; \frac{x}{t}, \frac{y}{1-t}\right) dt,$$

where $|x/t| < 1$, $|y/(1-t)| < 1$ along the contour.

(3·7) $$F_1(\alpha, \beta, \beta' ; \rho+\rho' ; x, y) = \frac{\Gamma(1-\rho)\Gamma(1-\rho')\Gamma(\rho+\rho')}{(2\pi i)^2}$$

$$\times \int (-t)^{\rho-1} (t-1)^{\rho'-1} F_1(\alpha, \beta, \beta' ; xt, yt)\, dt,$$

where $|xt| < 1$, $|yt| < 1$ along the contour.

(3·8) $$F_1(\alpha, \beta, \beta' ; \rho+\rho' ; x, y) = \frac{\Gamma(1-\rho)\Gamma(1-\rho')\Gamma(\rho+\rho')}{(2\pi i)^2}$$

$$\times \int (-t)^{\rho-1} (t-1)^{\rho'-1} F_2[\alpha, \beta, \beta' ; \rho, \rho' ; xt, y(1-t)]\, dt.$$

where $|xt| + |y(1-t)| < 1$, along the contour.

(3·9) $$F_2(\rho+\rho'-1, \beta, \beta' ; \gamma, \gamma' ; x, y) = \frac{\Gamma(\rho)\Gamma(\rho')\Gamma(2-\rho-\rho')}{(2\pi i)^2}$$

$$\times \int (-t)^{-\rho} (t-1)^{-\rho'} F_2\left(\rho, \beta, \beta' ; \gamma, \gamma' ; \frac{x}{t}, \frac{y}{t}\right) dt.$$

where $|x/t| + |y/t| < 1$, along the contour.

(3·10) $$F_2(\alpha, \beta, \beta'; \rho+\rho'; \gamma'; x, y) = \frac{\Gamma(1-\rho)\,\Gamma(1-\rho')\,\Gamma(\rho+\rho')}{(2\pi i)^2}$$
$$\times \int (-t)^{\rho-1}\,(t-1)^{\rho'-1}\,F_2(\alpha, \beta, \beta'; \rho, \gamma'; xt, y)\,dt,$$

where $|xt| + |y| < 1$ along the contour.

(3·11) $$F_3(\rho+\rho'-1, \alpha', \beta, \beta'; \gamma; x, y) = \frac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$$
$$\times \int (-t)^{-\rho}\,(t-1)^{-\rho'}\,F_3(\rho, \alpha', \beta, \beta'; \gamma; \frac{x}{t}, y)\,dt,$$

where $|x/t| < 1$, $|y| < 1$ along the contour.

(3·12) $$F_3(\alpha, \alpha', \beta, \beta'; \rho+\rho'; x, y) = \frac{\Gamma(1-\rho)\,\Gamma(1-\rho')\,\Gamma(\rho+\rho')}{(2\pi i)^2}$$
$$\times \int (-t)^{\rho-1}\,(t-1)^{\rho'-1}\,F_3(\alpha, \alpha', \beta\ \beta'; \rho; xt, yt)\,dt,$$

where $|xt| < 1$, $|yt| < 1$ along the contour.

(3·13) $$F_4(\rho+\rho'-1, \beta; \gamma, \gamma'; x, y) = \frac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$$
$$\times \int (-t)^{-\rho}\,(t-1)^{-\rho'}\,F_2(\beta, \rho', \rho; \gamma, \gamma'; \frac{x}{1-t}, \frac{y}{t})\,dt,$$

where $|x/(1-t)| + |y/t| < 1$ along the contour.

(3·14) $$F_4(\alpha, \beta; \rho+\rho'; \gamma'; x, y) = \frac{\Gamma(1-\rho)\,\Gamma(1-\rho')\,\Gamma(\rho+\rho')}{(2\pi i)^2}$$
$$\times \int (-t)^{\rho-1}\,(t-1)^{\rho'-1}\,F_4(\alpha, \beta; \rho, \gamma'; xt, y)\,dt,$$

where $|\sqrt{tx}| + |\sqrt{y}| < 1$ along the contour.

4. INTEGRAL REPRESENTATIONS OF SARAN'S HYPERGEOMETRIC FUNCTIONS OF THREE VARIABLES

If we employ the formula (3·9) in the expansion

$$F_E(\alpha_1, \alpha_1, \alpha_1\ \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z)$$
$$= \sum_{n=0}^{\infty} \frac{(\alpha_1\ n)\,(\beta_2\ n)}{(1, n)\ (\gamma_2, n)} F_2(\alpha_1+n, \beta_1, \beta_2+n; \gamma_1, \gamma_3; x, z)\,y^n,$$

and change the order of summation and integration, which is permissible when
$$\left|\frac{x}{t}\right| < R,\ \left|\frac{y}{t}\right| < S,\ \left|\frac{z}{t}\right| < T \text{ such that } R + (\sqrt{S}+\sqrt{T})^2 = 1$$
along the contour, we have

(4·1) $$F_E(\rho+\rho'-1, \rho+\rho'-1, \rho+\rho'-1, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) = \frac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$$
$$\times \int (-t)^{-\rho}\,(t-1)^{-\rho'}\,F_E(\rho, \rho, \rho, \beta_1, \beta_2, \beta_2; \gamma_1\ \gamma_2, \gamma_3; \frac{x}{t}, \frac{y}{t}, \frac{z}{t})\,dt$$

Similarly, we get

(4·2) $F_E(\rho+\rho'-1, \rho+\rho'-1, \rho+\rho'-1, \beta_1, \beta_2, \beta_2 ; \gamma_1, \gamma_2, \gamma_3 ; x, y, z) = \dfrac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$

$$\times \int (-t)^{-\rho}\,(t-1)^{-\rho'}\,F_K\left(\beta_1, \beta_2, \beta_2, \rho, \rho', \rho ; \gamma_1, \gamma_2, \gamma_3 ; \frac{x}{t}, \frac{y}{t}, \frac{z}{1-t}\right) dt,$$

where $\left|\frac{x}{t}\right| < R$, $\left|\frac{y}{t}\right| < S$, $\left|\frac{z}{1-t}\right| < T$ such that $T = (1-R)(1-S)$ along the contour;

(4·3) $F_F(\rho+\rho'-1, \rho+\rho'-1, \rho+\rho'-1, \beta_1, \beta_2, \beta_1 ; \gamma_1, \gamma_2, \gamma_2 ; x, y, z) = \dfrac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$

$$\times \int (-t)^{-\rho}\,(t-1)^{-\rho'}\,F_F\left(\rho, \rho, \rho, \beta_1, \beta_2, \beta_1 ; \gamma_1, \gamma_2, \gamma_2 ; \frac{x}{t}, \frac{y}{t}, \frac{z}{t}\right) dt,$$

where $|x/t| < R$, $|y/t| < S$, $|z/t| < T$ such that $RS = (1-S)(S-T)$ along the contour;

(4·4) $F_F(\rho+\rho'-1, \rho+\rho'-1, \rho+\rho'-1, \beta_1, \beta_2, \beta_1 ; \gamma_1, \gamma_2, \gamma_2 ; x, y, z) = \dfrac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$

$$\times \int (-t)^{-\rho}\,(t-1)^{-\rho'}\,F_M\left(\rho', \rho, \rho, \beta_1, \beta_2, \beta_1 ; \gamma_1, \gamma_2, \gamma_2 ; \frac{x}{1-t}, \frac{y}{t}, \frac{z}{t}\right) dt,$$

where $\left|\frac{x}{1-t}\right| < R$, $\left|\frac{y}{t}\right| < S$, $\left|\frac{z}{t}\right| < T$ such that $R + T - S = 1$ along the contour;

(4·5) $F_F(\rho+\rho'-1, \rho+\rho'-1, \rho+\rho'-1, \beta_1, \beta_2, \beta_1 ; \gamma_1, \gamma_2, \gamma_2 ; x, y, z) = \dfrac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$

$$\times \int (-t)^{-\rho}\,(t-1)^{-\rho'}\,F_P\left(\beta_1, \beta_2, \beta_1, \rho, \rho, \rho' ; \gamma_1, \gamma_2, \gamma_2 ; \frac{x}{t}, \frac{y}{t}, \frac{z}{1-t}\right) dt,$$

where $\left|\frac{x}{t}\right| < R$, $\left|\frac{y}{t}\right| < S$, $\left|\frac{z}{1-t}\right| < T$ such that $(ST-S-T)^2 = 4RST$ along the contour;

(4·6) $F_F(\rho+\rho'-1, \rho+\rho'-1, \rho+\rho'-1, \beta_1, \beta_2, \beta_1 ; \gamma_1, \gamma_2, \gamma_2 ; x, y, z) = \dfrac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho)}{(2\pi i)^2}$

$$\times \int (-t)^{-\rho}\,(t-1)^{-\rho'}\,F_R\left(\beta_1, \beta_2, \beta_1, \rho', \rho, \rho' ; \gamma_1, \gamma_2, \gamma_2 ; \frac{x}{1-t}, \frac{y}{t}, \frac{z}{1-t}\right) dt,$$

where $\left|\frac{x}{1-t}\right| < R$, $\left|\frac{y}{t}\right| < S$, $\left|\frac{z}{1-t}\right| < T$, such that $S(1-R) + (1-S)\,T = 0$ along the contour;

(4·7) $F_F(\alpha_1, \alpha_1, \alpha_1, \rho+\rho'-1, \beta_2, \rho+\rho'-1 ; \gamma_1, \gamma_2, \gamma_2 ; x, y, z) = \dfrac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$

$$\times \int (-t)^{-\rho}\,(t-1)^{-\rho'}\,F_G\left(\alpha_1, \alpha_1, \alpha_1, \rho', \beta, \rho ; \gamma_1, \gamma_2, \gamma_2 ; \frac{x}{1-t}, \frac{y}{t}, \frac{z}{t}\right) dt,$$

where $\left|\frac{x}{1-t}\right| < R$, $|y| < S$, $\left|\frac{z}{t}\right| < T$, such that $R+S=1=R+T$ along the contour;

(4·8) $$F_F(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \rho+\rho', \rho+\rho'; x, y, z) = \frac{\Gamma(1-\rho)\,\Gamma(1-\rho')\,\Gamma(\rho+\rho')}{(2\pi i)^2}$$
$$\times \int (-t)^{\rho-1}(t-1)^{\rho'-1} F_E[\alpha_1, \alpha_1, \alpha_1, \beta_2, \beta_1, \beta_1; \gamma, \rho, \rho'; y(1-t), x, zt]\, dt,$$

where $|y(1-t)| < R$, $|x| < S$, $|zt| < T$, such that $R+(S+T)=1$ along the contour;

(4·9) $$F_G(\rho+\rho'-1, \rho+\rho'-1, \rho+\rho'-1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \frac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$$
$$\times \int (-t)^{-\rho}(t-1)^{-\rho'} F_G\left(\rho, \rho, \rho, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_2; \frac{x}{t}, \frac{y}{t}, \frac{z}{t}\right) dt,$$

where $\left|\frac{x}{t}\right| < R$, $\left|\frac{y}{t}\right| < S$, $\left|\frac{z}{t}\right| < T$, such that $R+S=1=R+T$, along the contour;

(4·10) $$F_G(\rho+\rho'-1, \rho+\rho'-1, \rho+\rho'-1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \frac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$$
$$\times \int (-t)^{-\rho}(t-1)^{-\rho'} {}_2F_1\left(\beta_1, \rho'; \gamma_1; \frac{x}{1-t}\right) F_1\left(\rho, \beta_2, \beta_3; \gamma_2; \frac{y}{t}, \frac{z}{t}\right) dt,$$

where $\left|\frac{x}{1-t}\right| < 1$, $\left|\frac{y}{t}\right| < 1$, $\left|\frac{z}{t}\right| < 1$ along the contour;

(4·11) $$F_G(\rho+\rho'-1, \rho+\rho'-1, \rho+\rho'-1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \frac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$$
$$\times \int (-t)^{-\rho}(t-1)^{-\rho'} F_N\left(\beta_1, \beta_3, \beta_2, \rho, \rho', \rho; \gamma_1, \gamma_2, \gamma_2; \frac{x}{t}, \frac{y}{1-t}, \frac{z}{t}\right) dt,$$

where $\left|\frac{x}{t}\right| < R$, $\left|\frac{y}{t}\right| < S$, $\left|\frac{z}{t}\right| < T$, such that $(1-R)\,S + (1-S)\,T = 0$ along the contour;

(4·12) $$F_G(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3; \gamma_1, \rho+\rho', \rho+\rho'; x, y, z) = \frac{\Gamma(1-\rho)\,\Gamma(1-\rho')\,\Gamma(\rho+\rho')}{(2\pi i)^2}$$
$$\times \int (-t)^{\rho-1}(t-1)^{\rho'-1} F_A(\alpha_1, \beta_1, \beta_2, \beta_3; \gamma_1, \rho, \rho'; x, yt, z(1-t))\, dt,$$

where $|x| < R$, $|yt| < S$, $|z(1-t)| < T$, such that $R+S+T=1$ along the contour;

(4·13) $$F_K(\alpha_1, \rho+\rho'-1, \rho+\rho'-1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; x, y, z) = \frac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$$
$$\times \int (-t)^{-\rho}(t-1)^{-\rho'} F_K\left(\alpha_1, \rho, \rho, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; x, \frac{y}{t}, \frac{z}{t}\right) dt,$$

where $|x| < R$, $\left|\frac{y}{t}\right| < S$, $\left|\frac{z}{t}\right| < T$, such that $(1-R)(1-S) = T$, along the contour;

(4·14) $F_M(\alpha_1, \rho+\rho'-1, \rho+\rho'-1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \dfrac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$

$$\times \int (-t)^{-\rho}\,(t-1)^{-\rho'}\, F_M\!\left(\alpha_1, \rho, \rho, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, \frac{y}{t}, \frac{z}{t}\right) dt,$$

where $|x| < R$, $\left|\dfrac{y}{t}\right| < S$, $\left|\dfrac{z}{t}\right| < T$, such that $R + T = 1 = S$, along the contour;

(4·15) $F_M(\alpha_1, \rho+\rho'-1, \rho+\rho'-1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \dfrac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$

$$\times \int (-t)^{-\rho}\,(t-1)^{-\rho'}\, F_N\!\left(\alpha_1, \rho, \rho', \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, \frac{y}{t}, \frac{z}{1-t}\right) dt,$$

where $|x| < R$, $\left|\dfrac{y}{t}\right| < S$, $\left|\dfrac{z}{1-t}\right| < T$, such that $(1-R)S + (1-S)T = 0$ along the contour;

(4·16) $F_M(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_2; \gamma_1, \rho+\rho', \rho+\rho'; x, y, z) = \dfrac{\Gamma(1-\rho)\,\Gamma(1-\rho')\,\Gamma(\rho+\rho')}{(2\pi i)^2}$

Hence

$$\times \int (-t)^{\rho-1}\,(t-1)^{\rho'-1}\, F_K\,[\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \rho, \rho'; x, yt, z(1-t)]\, dt,$$

where $|x| < R$, $|yt| < S$, $|z(1-t)| < T$, such that $(1-R)(1-S) = T$ along the contour;

(4·17) $F_N(\alpha_1, \alpha_2, \alpha_3; \rho+\rho'-1, \beta_2, \rho+\rho'-1; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \dfrac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$

$$\times \int (-t)^{-\rho}\,(t-1)^{-\rho'}\, F_N\!\left(\alpha_1, \alpha_2, \alpha_3, \rho, \beta_2, \rho; \gamma_1, \gamma_2, \gamma_2; \frac{x}{t}, y, \frac{z}{t}\right) dt,$$

where $\left|\dfrac{x}{t}\right| < R$, $|y| < S$, $\left|\dfrac{z}{t}\right| < T$, such that $(1-R)S + (1-S)T = 0$ along the contour;

(4·18) $F_N(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_1; \gamma_1, \rho+\rho', \rho+\rho'; x, y, z) = \dfrac{\Gamma(1-\rho)\,\Gamma(1-\rho')\,\Gamma(\rho+\rho')}{(2\pi i)^2}$

$$\times \int (-t)^{\rho-1}\,(t-1)^{\rho'-1}\, {}_2F_1[\alpha_2, \beta_2; \rho'; y(1-t)]\, F_2(\beta_1, \alpha_1, \alpha_3; \gamma_1, \rho; x, zt)\, dt,$$

where $|y(1-t)| < 1$, and $|x| + |zt| < 1$ along the contour;

(4·19) $F_P(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \rho+\rho', \rho+\rho', x, y, z) = \dfrac{\Gamma(1-\rho)\,\Gamma(1-\rho')\,\Gamma(\rho+\rho')}{(2\pi i)^2}$

$$\times \int (-t)^{\rho-1}\,(t-1)^{\rho'-1}\, F_P(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \rho, \rho; x, yt, zt)\, dt,$$

where $|x| < R$, $|yt| < S$, $|zt| < T$, such that $(RT - S - T)^2 = 4RST$ along the contour;

$$(4{\cdot}20)\quad F_P(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \rho+\rho', \rho+\rho'; x, y, z) = \frac{\Gamma(1-\rho)\,\Gamma(1-\rho')\,\Gamma(\rho+\rho')}{(2\pi i)^2}$$

$$\times \int (-t)^{\rho-1}\,(t-1)^{\rho'-1}\, F_K[\beta_2, \beta_1, \beta_1, \alpha_1, \alpha_2, \alpha_1; \rho, \rho', \gamma_1; zt, y(1-t), x]\, dt,$$

where $|zt| < R$, $|y(1-t)| < S$, $|x| < T$, such that $(1-R)(1-S) = T$ along the contour;

$$(4{\cdot}21)\quad F_R(\rho+\rho'-1, \alpha_2, \rho+\rho'-1, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_2; x, y, z) = \frac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$$

$$\times \int (-t)^{-\rho}\,(t-1)^{-\rho'}\, F_N(\rho', \alpha_2, \rho, \beta_1, \beta_2, \beta_1, \gamma_1, \gamma_2, \gamma_2; \frac{x}{1-t}, y, \frac{z}{t})\, dt,$$

where $\left|\frac{x}{1-t}\right| < R$, $|y| < S$, $\left|\frac{z}{t}\right| < T$, such that $(1-R)S + (1-S)T = 0$ along the contour;

$$(4{\cdot}22)\quad F_R(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \rho+\rho', \rho+\rho'; x, y, z) = \frac{\Gamma(1-\rho)\,\Gamma(1-\rho')\,\Gamma(\rho+\rho')}{(2\pi i)^2}$$

$$\times \int (-t)^{\rho-1}\,(t-1)^{\rho'-1}\, F_R(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_2, \beta_1; \gamma_1, \rho, \rho; x, yt, zt)\, dt,$$

where $|x| < R$, $|yt| < S$, $|zt| < T$, such that $S(1-\sqrt{R})^2 + (1-S)T = 0$ along the contour;

$$(4{\cdot}23)\quad F_S(\alpha_1, \rho+\rho'-1, \rho+\rho'-1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_1, \gamma_1; x, y, z) = \frac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$$

$$\times \int (-t)^{-\rho}\,(t-1)^{-\rho'}\, F_S(\alpha_1, \rho, \rho, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_1, \gamma_1; x, \frac{y}{t}, \frac{z}{t})\, dt,$$

where $|x| < R$, $\left|\frac{y}{t}\right| < S$, $\left|\frac{z}{t}\right| < T$, such that $R + S = RS$, $S = T$ along the contour;

$$(4{\cdot}24)\quad F_S(\alpha_1, \rho+\rho'-1, \rho+\rho'-1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_1, \gamma_1; x, y, z) = \frac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$$

$$\times \int (-t)^{-\rho}\,(t-1)^{-\rho'}\, F_B(\alpha_1, \rho, \rho', \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_1, \gamma_1; x, \frac{y}{t}, \frac{z}{t})\, dt,$$

where $|x| < 1$, $\left|\frac{y}{t}\right| < 1$, $\left|\frac{z}{t}\right| < 1$ along the contour;

$$(4{\cdot}25)\quad F_S(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3; \rho+\rho', \rho+\rho', \rho+\rho'; x, y, z) = \frac{\Gamma(1-\rho)\,\Gamma(1-\rho')\,\Gamma(\rho+\rho')}{(2\pi i)^2}$$

$$\times \int (-t)^{\rho-1}\,(t-1)^{\rho'-1}\, F_S(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3; \rho, \rho, \rho; xt, yt, zt)\, dt,$$

where $|xt| < R$, $|yt| < S$, $|zt| < T$, such that $R + S = RS$, $S = T$ along the contour;

(4·26) $F_S(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3; \rho+\rho', \rho+\rho', \rho+\rho'; x, y, z) = \dfrac{\Gamma(1-\rho)\,\Gamma(1-\rho')\,\Gamma(\rho+\rho')}{(2\pi i)^2}$

$$\times \int (-t)^{\rho-1}\,(t-1)^{\rho'-1}\,F_N(\beta_3, \beta_1, \beta_2, \alpha_2, \alpha_1, \alpha_2; \rho', \rho, \rho; z(1-t), yt, xt)\,dt,$$

where $|z(1-t)| < R$, $|yt| < S$, $|xt| < T$, such that $(1-R)S + (1-S)T = 0$ along the contour;

(4·27) $F_T(\alpha_1, \alpha_2, \alpha_2, \rho+\rho'-1, \beta_2, \rho+\rho'-1; \gamma_1, \gamma_1, \gamma_1; x, y, z) = \dfrac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$

$$\times \int (-t)^{-\rho}\,(t-1)^{-\rho'}\,F_T(\alpha_1, \alpha_2, \alpha_2, \rho, \beta_2, \rho; \gamma_1, \gamma_1, \gamma_1; \frac{x}{t}, y, \frac{z}{t})\,dt,$$

where $\left|\frac{x}{t}\right| < R$, $|y| < S$, $\left|\frac{z}{t}\right| < T$, such that $R = RS + S = T$, along the contour;

(4·28) $F_T(\alpha_1, \alpha_2, \alpha_2, \rho+\rho'-1, \beta_2, \rho+\rho'-1; \gamma_1, \gamma_1, \gamma_1; x, y, z) = \dfrac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$

$$\times \int (-t)^{-\rho}\,(t-1)^{-\rho'}\,F_S(\alpha_1, \alpha_1, \alpha_2, \rho, \beta_2, \rho; \gamma_1, \gamma_1, \gamma_1; \frac{x}{t}, y, \frac{z}{1-t})\,dt,$$

where $\left|\frac{x}{t}\right| < R$, $|y| < S$, $\left|\frac{z}{1-t}\right| < T$, such that $R + S = RS$, $S = T$ along the contour;

and

(4·29) $F_T(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \rho+\rho', \rho+\rho', \rho+\rho', x, y, z) = \dfrac{\Gamma(1-\rho)\,\Gamma(1-\rho')\,\Gamma(\rho+\rho')}{(2\pi i)^2}$

$$\times \int (-t)^{\rho-1}\,(t-1)^{\rho'-1}\,F_P(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \rho, \rho, \rho'; xt, yt, z(1-t))\,dt,$$

where $|xt| < R$, $|yt| < S$, $|z(1-t)| < T$ such that $(RT - S - T)^2 = 4RST$ along the contour.

5. CONTOR INTEGRALS ASSOCIATED WITH SRIVASTAVA'S H_A, H_B AND H_C

We now use the formula (3·5) in

$$H_A(\alpha, \beta, \beta'; \gamma, \gamma'; x, y, z) = \sum_{m=0}^{\infty} \frac{(\alpha, m)\,(\beta, m)}{(1, m)\,(\gamma, m)}\,F_1(\beta', \beta+m, \alpha+m; \gamma'; y, z)\,x^m,$$

and reverse the order of summation and integration which is readily justified when $|x| < R$, $\left|\frac{y}{t}\right| < S$, $\left|\frac{z}{t}\right| < T$, such that $R + S + T = 1 + ST$ along the contour; and we find that

(5·1) $H_A(\alpha, \beta, \rho+\rho'-1; \gamma, \gamma'; x, y) = \dfrac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$

$$\times \int (-t)^{-\rho}\,(t-1)^{-\rho'}\,H_A(\alpha, \beta, \rho; \gamma, \gamma'; x, \frac{y}{t}, \frac{z}{t})\,dt,$$

under the conditions stated earlier.

In a similar way we get

(5·2) $$HA(\alpha,\ \beta,\ \rho+\rho'-1\ ;\ \gamma,\ \gamma'\ ;\ x,\ y,\ z) = \frac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$$

$$\times \int (-t)^{-\rho}\,(t-1)^{-\rho'}\,F_P(\alpha,\ \rho,\ \alpha,\ \beta,\ \beta,\ \rho'\ ;\ \gamma,\ \gamma',\ \gamma'\ ;\ x,\ \frac{y}{t},\ \frac{z}{1-t})\ dt,$$

where $|\,x\,| < R$, $|\,yt\,| < S$, $|\,zt\,| < T$, such that $(RT - S - T)^2 = 4RST$ along the contour;

(5·3) $$HA(\alpha,\ \beta,\ \beta'\ ;\ \gamma,\ \rho+\rho'\ ;\ x,\ y,\ z) = \frac{\Gamma(1-\rho)\,\Gamma(1-\rho')\,\Gamma(\rho+\rho')}{(2\pi i)^2}$$

$$\times \int (-t)^{\rho-1}\,(t-1)^{\rho'-2}\,HA(\alpha,\ \beta,\beta'\ ;\ \gamma,\ \rho\ ;\ x,\ yt,\ zt)\ dt,$$

where $|\,x\,| < R$, $|\,yt\,| < S$, $|\,zt\,| < T$, such that $R + S + T = 1 + ST$, along the contour;

(5·4) $$HA(\alpha,\ ,\ \beta'\ ;\ \gamma,\ \rho+\rho'\ ;\ x,\ y,\ z) = \frac{\Gamma(1-\rho)\,\Gamma(1-\rho')\,\Gamma(\rho+\rho')}{(2\pi i)^2}$$

$$\times \int (-t)^{\rho-1}\,(t-1)^{\rho'-1}\,HB[\alpha,\ \beta,\ \beta'\ ;\ \gamma,\ \rho,\ \rho'\ ;\ x,\ yt,\ z(1-t)]\ dt,$$

where $|\,x\,| < R$, $|\,yt\,| < S$, $|\,z(1-t)\,| < T$, such that $R + S + T + 2\sqrt{RST} = 1$ along the contour;

(5·5) $$HB(\alpha,\ \beta,\ \rho+\rho'-1\ ;\ \gamma_1,\ \gamma_2,\ \gamma_3\ ;\ x,\ y,\ z) = \frac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$$

$$\times \int (-t)^{-\rho}\,(t-1)^{-\rho'}\,HB(\alpha,\ \beta,\ \rho\ ;\ \gamma_1,\ \gamma_2\ \gamma_3\ ;\ x,\ \frac{y}{t},\ \frac{z}{t})\ dt,$$

where $|\,x\,| < R$, $|\,\frac{y}{t}\,| < S$, $|\,\frac{z}{t}\,| < T$, such that $R + S + T + 2\sqrt{RST} = 1$ along the contour;

(5·6) $$HC(\alpha,\ \beta,\ \rho+\rho'-1\ ;\ \gamma\ ;\ x,\ y,\ z) = \frac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$$

$$\times \int (-t)^{-\rho}\,(t-1)^{-\rho'}\,HC(\alpha,\ \beta,\ \rho\ ;\ \gamma\ ;\ x,\ \frac{y}{t},\ \frac{z}{t})\ dt,$$

where $|\,x\,| < 1$, $|\,\frac{y}{t}\,| < 1$, $|\,\frac{z}{t}\,| < 1$, along the contour;

(5·7) $$HC(\alpha,\ \beta,\ \rho'+\rho-1\ ;\ \gamma\ ;\ x,\ y,\ z) = \frac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(2-\rho-\rho')}{(2\pi i)^2}$$

$$\times \int (-t)^{-\rho}\,(t-1)^{-\rho'}\,F_T(\rho,\ \alpha,\ \alpha,\ \beta,\ \rho',\ \alpha\ ;\ \gamma,\ \gamma,\ \gamma\ ;\ \frac{y}{t},\ \frac{z}{1-t},\ x)\ dt,$$

where $|\,x\,| < R$, $|\,\frac{y}{t}\,| < S$, $|\,\frac{z}{1-t}\,| < T$, such that $R = S - ST + T$ along the contour;

$$(5{\cdot}8) \qquad H_C(\alpha, \beta, \beta' : \rho+\rho' ; x, y, z) = \frac{\Gamma(1-\rho)\,\Gamma(1-\rho')\,\Gamma(\rho+\rho')}{(2\pi i)^2}$$

$$\times \int (-t)^{\rho-1} (t-1)^{\rho'-1}\, H\,(\alpha, \beta, \beta' : \rho ; xt, yt, zt)\, dt,$$

where $|xt| < R$, $|yt| < S$, $|zt| < T$, such that $R + S + T = 1$, along the contour;

$$(5{\cdot}9) \qquad H_C(\alpha, \beta, \beta' ; \rho+\rho' ; x, y, z) = \frac{\Gamma(1-\rho)\,\Gamma(1-\rho')\,\Gamma(\rho+\rho')}{(2\pi i)^2}$$

$$\times \int (-t)^{\rho-1} (t-1)^{\rho'-1}\, H_A[\beta', \beta, \alpha : \rho, \rho' ; yt, x(1-t), z(1-t)]\, dt,$$

whare $|yt| < R$, $|x(1-t)| < S$, $|z(1-t)| < T$, such that $R + S + T = 1 + ST$ along the contour;

and

$$(5{\cdot}10) \qquad H_C(\alpha, \beta, \beta' ; \rho+\rho' ; x, y, z) = \frac{\Gamma(1-\rho)\,\Gamma(1-\rho')\,\Gamma(\rho+\rho')}{(2\pi i)^2}$$

$$\times \int (-t)^{\rho-1} (t-1)^{\rho'-1}\, H_A[\alpha, \beta, \beta' ; \rho, \rho', xt, yt, z(1-t)]\, dt,$$

where $|xt| < R$, $|yt| < S$, $|z(1-t)| < T$, such that $R + S + T = 1 + ST$ along the contour;

6. INTEGRATION OF SYSTEMS OF HYPERGEOMETRIC DIFFERENTIAL EQUATIONS

The integral (4·4) suggests that

$$(6{\cdot}1) \qquad W = \int_C t^{-\rho}\, (1-t)^{-\rho'}\, f\left(\frac{x}{1-t}, \frac{y}{t}, \frac{z}{t}\right) dt,$$

should be a solution of the system of partial differential equations

$$(6{\cdot}2) \qquad \begin{aligned} &(i)\ [\theta(\theta+\gamma_1-1) - x(\theta+\phi+\psi+\rho+\rho'-1)(\theta+\phi+\beta_1)]\, W = 0 \\ &(ii)\ [\phi(\phi+\psi+\gamma_2-1) - y(\theta+\phi+\psi+\rho+\rho'-1)(\phi+\beta_2)]\, W = 0 \\ &(iii)\ [\psi(\psi+\phi+\gamma_2-1) - z(\theta+\phi+\psi+\rho+\rho'-1)(\theta+\psi+\beta_1)]\, W = 0 \end{aligned}$$

associated with Saran's F_F, where C is some closed contour in the t-plane $\theta, \phi, \psi \equiv x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, z\frac{\partial}{\partial z}$ and $f(u, v, w)$ is an integral of the differential system

$$(6{\cdot}3) \qquad \begin{aligned} &(i)\ [\theta_1(\theta_1+\gamma_1-1) - u(\theta_1+\rho')(\theta_1+\phi_1+\beta_1)]\, f(u, v, w) = 0 \\ &(ii)\ [\phi_1(\phi_1+\psi_1+\gamma_2-1) - v(\phi_1+\psi_1+\rho)(\phi_1+\beta_2)]\, f(u, v, w) = 0 \\ &(iii)\ [\psi_1(\phi_1+\psi_1+\gamma_2-1) - w(\psi_1+\phi_1+\rho)(\theta_1+\psi_1+\beta_1)]\, f(u, v, w) = 0 \end{aligned}$$

where $\theta_1, \phi_1, \psi_1 \equiv u\frac{\partial}{\partial u}, v\frac{\partial}{\partial v}, w\frac{\partial}{\partial w}$ with $u = \frac{x}{1-t}$, $v = \frac{y}{t}$, $w = \frac{z}{t}$

Now denoting the differential system (6·2) by $L_1(w) = 0$, $L_2(w) = 0$ and $L_3(w) = 0$ respectively, we observe that

$$L_1(W) = \int_C t^{-\rho} (1-t)^{-\rho'} [\theta(\theta+\gamma_1-1) - x(\theta+\phi+\psi+\rho+\rho'-1)(\theta+\psi+\beta_1)] f(u, v, w)\, dt$$

and in view of (6·3) (i) this gives us

$$L_1(W) = x \int_C t^{-\rho} (1-t)^{-\rho'} [t(\theta_1+\rho') - (1-t)(\phi_1+\psi_1+\rho-1)] (\theta_1+\psi_1+\beta_1) f(u, v, w)\, dt$$

Now for the function $f(u, v, w)$ it is easily seen that

$t(1-t)\,\partial/\partial t = t\theta_1 - (1-t)\,\phi_1 - (1-t)\,\psi_1$, and therefore

(6·4) $$L_1(W) = x\int_C d\left[t^{-\rho+1}(1-t)^{-\rho'} \left(u\frac{\partial f}{\partial u} + w\frac{\partial f}{\partial w} + \beta_1 f\right)\right],$$

Similarly, we have

(6·5) $$L_2(W) = y\int_C d\left[t^{-\rho} (1-t)^{-\rho'+1} \left(v\frac{\partial f}{\partial v} + \beta_2 f\right)\right],$$

and

(6·6) $$L_3(W) = z\int_C d\left[t^{-\rho} (1-t)^{-\rho'+1} \left(u\frac{\partial f}{\partial u} + w\frac{\partial f}{\partial w} + \beta_1 f\right)\right].$$

Equations (6·4), (6·5) and (6·6) exhibit the fact that (6·1) is certainly a solution of (6·2) whenever c is a closed contour or else an open contour at the two ends of which each of

$$\left[t^{-\rho+1} (1-t)^{-\rho'} \left(u\frac{\partial f}{\partial u} + w\frac{\partial f}{\partial w} + \beta_1 f\right)\right],$$

$$\left[t^{-\rho} (1-t)^{-\rho'+1} \left(v\frac{\partial f}{\partial v} + \beta_2 f\right)\right],$$

and

$$\left[t^{-\rho} (1-t)^{-\rho'+1} \left(u\frac{\partial f}{\partial u} + w\frac{\partial f}{\partial w} + \beta_1 f\right)\right]$$

vanishes, where $f \equiv f(u, v, w)$ is any solution of 6·3).

Similarly, the general solution of the hypergeometric differential system [8, (4·1)]

(6·7) $$\begin{cases} [\theta(\theta+\gamma-1) - x(\theta+\psi+\alpha)(\theta+\phi+\beta)]\, W = 0 \\ [\phi(\phi+\psi+\gamma'-1) - y(\theta+\phi+\beta)(\phi+\psi+\rho+\rho'-1)]\, W = 0 \\ [\psi(\phi+\psi+\gamma'-1) - z(\theta+\psi+\alpha)(\theta+\psi+\rho+\rho'-1)]\, W = 0 \end{cases}$$

associated with Srivastava's H_A, is given by

(6·8) $$W = \int_C t^{-\rho} (1-t)^{-\rho'} g\left(x, \frac{y}{t}, \frac{z}{1-t}\right) dt$$

provided that c is a closed contour or else an open one at the two ends of which both

$$\left[t^{-\rho} (1-t)^{-\rho'+1} \left(u \frac{\partial g}{\partial u} + v \frac{\partial g}{\partial v} + \beta g\right) \right]$$

and

$$\left[t^{-\rho+1} (1-t)^{-\rho'} \left(u \frac{\partial g}{\partial u} + w \frac{\partial g}{\partial w} + \alpha g\right) \right]$$

vanish, where $g = g(u, v, w)$ is any solution of the partial differential equations

$$(6{\cdot}9) \qquad \begin{cases} [\theta_1 (\theta_1 + \gamma - 1) - u (\theta_1 + \phi_1 + \alpha)(\theta_1 + \phi_1 + \beta)]\, g(u, v, w) = 0 \\ [\phi_1 (\phi_1 + \psi_1 + \gamma' - 1) - v (\phi_1 + \rho)(\phi_1 + \theta_1 + \beta)]\, g(u, v, w) = 0 \\ [\psi_1 (\phi_1 + \psi_1 + \gamma' - 1) - w (\theta_1 + \psi_1 + \alpha)(\psi_1 + \rho')]\, g(u, v, w) = 0 \end{cases}$$

where $\theta_1, \phi_1, \psi_1 = u \frac{\partial}{\partial u}, v \frac{\partial}{\partial v}, w \frac{\partial}{\partial w}$ with $u = x$, $v = \frac{y}{t}$ and $w = \frac{z}{1-t}$.

To illustrate this method of integration of systems partial equations we consider a branch of the differential system (6·9) namely

$$u^{1-\gamma} F_P(\alpha+1-\gamma, \rho, \alpha+1-\gamma, \beta-\gamma+1, \beta-\gamma+1, \rho'; 2-\gamma, \gamma', \gamma'; u, v, w),$$

substituting for $g(u, v, w)$ in the contour integral (6·8) and we obtain a solution of (6·7) in the form

$$W = A\, x^{1-\gamma} H_A(\alpha+1-\gamma, \beta-\gamma+1, \rho+\rho'-1; 2-\gamma, \gamma'; x, y, z)$$

valid in the vicinity of the origin, A being an arbitrary constant.

7. TRANSFORMATIONS AND REDUCIBLE CASES

In this section we illustrate how our results of the preceding sections can be used to derive certain linear hypergeometric transformations associated with the functions represented by them.

We start with the contour integral representation of F_E namely

$$F_E(\rho+\rho'-1, \rho+\rho'-1, \rho+\rho'-1, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z) = \frac{\Gamma(\rho)\, \Gamma(\rho')\, \Gamma(2-\rho-\rho')}{(2\pi i)^2}$$

$$\times \int (-t)^{-\rho} (t-1)^{-\rho'} F_K\left(\beta_1, \beta_2, \beta_2, \rho, \rho', \rho; \gamma_1, \gamma_2, \gamma_3; \frac{x}{t}, \frac{y}{t}, \frac{z}{1-t}\right) dt$$

where $\left|\frac{x}{t}\right| < R$, $\left|\frac{y}{t}\right| < S$ and $\left|\frac{z}{1-t}\right| < T$, such that $(1-R)(1-S) = T$ along the contour, apply the formula ([5], p. 88; *see* also [4], p. 113)

$$F_K(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; x, y, z)$$

$$= (1-x)^{-\beta_1}\, F_K\left(\gamma_1-\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; \frac{x}{x-1}, y, \frac{z}{1-x}\right);$$

and then introduce $\frac{x-t}{x-1}$ as a new variable of integration.

We thus obtain the transformation

$$(7{\cdot}1) \qquad F_E(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_1 ; \gamma_1, \gamma_2, \gamma_3 ; x, y, z)$$

$$= (1-x)^{-\alpha_1} \; F_E(\alpha_1, \alpha_1, \alpha_1, \gamma_1-\beta_1, \beta_2, \beta_2 ; \gamma_1, \gamma_2, \gamma_3 ; \frac{x}{x-1}, \frac{y}{1-x}, \frac{z}{1-x}),$$

due to Saran [6, p. 87].

In a similar way, by virtue of the known formula [6, p. 89]

$$F_N(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_1 ; \gamma_1, \gamma_2, \gamma_2 ; x, y, z)$$

$$= (1-x)^{-\beta_1} \; F_N(\gamma_1-\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_1 ; \gamma_1, \gamma_2, \gamma_2 ; \frac{x}{x-1}, y, \frac{z}{1-x}),$$

the integral (4·11) gives us the transformation

$$(7{\cdot}2) \qquad F_G(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3 ; \gamma_1, \gamma_2, \gamma_3 ; x, y, z)$$

$$= (1-x)^{-\alpha_1} \; F_G(\alpha_1, \alpha_1, \alpha_1, \gamma_1-\beta_1, \beta_2, \beta_3 ; \gamma_1, \gamma_2, \gamma_3 ; \frac{x}{x-1}, \frac{y}{1-x}, \frac{z}{1-x}),$$

which holds wherever the triple series converge simultaneously [6, p. 87].

As an implication of Srivastava's transformation [7, p. 69]

$$F_P(\alpha_1, \alpha_2, \alpha_1, \beta_1, \beta_1, \beta_2 ; \gamma_1, \gamma_2, \gamma_2 ; x, y, z)$$

$$= (1-y)^{-\beta_1} \; H_A[\alpha_1, \beta_1, \beta_2 ; \gamma_1, \gamma_2 ; \frac{x}{1-y}, \frac{y}{y-1}, z]$$

where $\gamma_2 = \alpha_2 + \beta_2$, from the integral (5·2) it can also be shown that

$$(7{\cdot}3) \qquad H_A(\alpha, \beta, \beta' ; \gamma, \gamma' ; x, y, z)$$

$$= (1-y)^{-\beta} \; (1-z)^{-\alpha} H_A(\alpha, \beta, \gamma'-\beta' ; \gamma, \gamma' ; \frac{x}{(y-1)(z-1)}, \frac{y}{y-1}, \frac{z}{z-1}),$$

a formula proved earlier by Srivastava (*see* [8] and [9]) in a different way.

Next we make use of the reduction formula

$$F_K(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1 ; \alpha_1, \gamma_2, \gamma_3 ; x, y, z)$$

$$= (1-x)^{-\beta_1} \; F_2[\alpha_2, \beta_2, \beta_1 ; \gamma_2, \gamma_3 ; y, \frac{z}{1-x}]$$

given recently by Srivastava [7, p. 71], and from the integral representation (4·16) we find that

$$F_M(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1 ; \gamma_1, \rho+\rho', \rho+\rho' ; x, y, z) = \frac{\Gamma(1-\rho)\,\Gamma(1-\rho')\,\Gamma(\rho+\rho')}{(2\pi i)^2}$$

$$\times \; (1-x)^{-\beta_1} \int (-t)^{\rho-1} (t-1)^{\rho'-1} F_2{}'[\alpha_2, \beta_2, \beta_1 ; \rho, \rho' ; yt, \frac{z(1-t)}{(1-x)}] \, dt,$$

where $\gamma_1 = \alpha_1$, etc.

On evaluating the last contour integral by means of our formula (5·8) we get the interesting reduction [7 (5·7)]

$$(7\cdot4) \qquad F_M(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1, \gamma_1, \gamma_2, \gamma_2; x, y, z)$$

$$= (1-x)^{-\beta_1} F_1\left[\alpha_2, \beta_2, \beta_1; \gamma_1; y, \frac{z}{1-x}\right]$$

Similar consequences of our integral (4·2) is the known formula

$$(7\cdot5) \qquad F_E(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z)$$

$$= (1-x)^{-\alpha_1} F_4\left[\alpha_1, \beta_2; \gamma_2, \gamma_3; \frac{y}{1-x}, \frac{z}{1-x}\right]$$

which holds inside the common domain of absolute convergence of the two series [7, p. 71]

The above methods when applied to the remaining contour integrals of the preceding sections will yield a number of transformations and cases of reducibility of the various hypergeometric functions of three variables.

REFERENCES

1. Appell P. and Kampé de Fériet J. Fonctions hypergéométriques et hypersphériques, Polynomés d'Hermite (*Paris*, 1926).
2. Erdélyi, A. *et al.* Higher transcendental functions *McGraw Hill*, Vol. I, (1953).
3. Erdélyi, A. Transformations of hypergeometric functions of two variables *Proc. Roy. Soc. Edinburgh*, Sec. A 62 : 378–385, (1948).
4. Pandey, R. C. On certain hypergeometric transformations. *J. Math Mech* 12 : 113–118, (1963).
5. Saran, S. Integrals associated with hypergeometric functions of three variables. *Proc. Nat Inst. Sci., India*, Sec. A 21 : 83–90, (1955).
6. Saran, S Hypergeometric functions of three variables *Ganita* 5 : 77–91, (1954).
7. Srivastava, H. M. On transformations of certain hypergeometric functions of three variables. *Publictions Math*, 12 : 65–74, (1965
8. Srivastava, H. M Hypergeometric functions of three variables. *Ganita*, 15 (2) : (1964).
9. Srivastava, H. M. Some integrals representing triple hypergeometric functions (*in course of publication*).
10. Whittaker, E. T. and Watson, G. N. A course of Modern Analysis (*Cambridge*, 1952)

A DEFINITE INTEGRAL INVOLVING MEIJER'S G-FUNCTION

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ABSTRACT

In this note a definite integral has been evaluated by making use of a theorem given by Hari Shanker, which generalises the result given earlier by Saxena.

1. INTRODUCTION

The aim of this note is to evaluate a definite integral involving Meijer's G-function, in which the argument of the G-function contains a factor of the type $\left(\dfrac{n\cos^2\theta}{p+a\cos^2\theta+b\sin^2\theta}\right)^n$, where θ is the variable of the integration and n is a positive integer.

As usual the conventional notation $\phi(p) \doteqdot h(t)$, will be used to denote the classical Laplace transform

$$\phi(p) = p\int_0^\infty e^{-pt}\, h(t)\, dt, \tag{1}$$

provided that the integral is convergent and $R(p) > 0$.

The symbol $\triangle(n\,;\alpha)$ denotes the set of parameters

$$\frac{\alpha}{n}, \frac{\alpha+1}{n}, \frac{\alpha+2}{n}, \ldots\ldots, \frac{\alpha+n-1}{n}.$$

2. Here we shall make use of a theorem given by a Hari Shanker [2, p. 44] which states that if

$$f_1(p) \doteqdot h_1(t),$$

and

$$f_2(p) \doteqdot h_2(t),$$

then

$$\int_0^{\pi/2} F(p,\cos^2\theta, \sin^2\theta)\, \sin 2\theta\, d\theta = p^{-1} f_1(p) f_2(p), \tag{2}$$

where

$$t\, h_1(xt)\, h_2(yt) \doteqdot F(p, x, y)$$

We shall also make use of a well known property of operational calculus that if $h(t) \doteqdot \phi(p)$, then

$$e^{-at}\, h(t) \doteqdot \frac{p}{p+a}\, \phi(p+a), \text{ where } R(p+a) > 0.$$

3. APPLICATION

Start with Saxena [3, p. 402 (11)]

$$h_1(t)=e^{-at}\,t^{-\rho}\;G^{\alpha,\beta}_{\gamma,\delta}\left(t^n\left|\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_\delta\end{matrix}\right.\right)$$

$$\doteqdot\frac{p(2\pi)^{\frac{1}{2}(1-n)}}{n^{\rho-\frac{1}{2}}(p+a)^{1-\rho}}\;G^{\alpha,\beta+n}_{\gamma+n,\delta}\left(\frac{n^n}{(p+a)^n}\left|\begin{matrix}\wedge(n\,;\rho),a_1,\dots,a_r\\ b_1,\dots,b_\delta\end{matrix}\right.\right)$$

$= f_1(p)$, where $R(p+a)>0$, $R(1-\rho+n\,b_h)>0$, $(h=1,2,\dots\alpha)$,

$0\leqslant\beta\leqslant\gamma<\infty$ and $1\leqslant\alpha\leqslant\delta$

and Erdelyi [1, p. 137 (1)]

$$h_2(t)=e^{-bt}\,t^{\mu-1}$$

$$\doteqdot\Gamma(\mu)\,p(p+b)^{-\mu}$$

$= f_2(p)$, where $R(p+b)>0$.

Next, again Saxena [3, p. 402 (11)]

$$t\,h_1(xt)\,h_2(yt)=e^{-(ax+by)t}\,t^{\mu-\rho}\,G^{\alpha,\beta}_{\gamma,\delta}\left(x^n t^n\left|\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_\delta\end{matrix}\right.\right)$$

$$\doteqdot\frac{py^{\mu-1}(2\pi)^{\frac{1}{2}(1-n)}\,n^{\frac{1}{2}-\rho+\mu}}{x^\rho\,(p+ax+by)^{1+\mu-\rho}}\;G^{\alpha,\beta+n}_{\gamma+n,\delta}\left(\frac{x^n n^n}{(p+ax+by)^n}\left|\begin{matrix}\wedge(n\,;\rho-\mu),a_1,\dots,a_r\\ b_1,\dots,b_\delta\end{matrix}\right.\right)$$

$= F(p,x,y)$, where $R(p+ax+by)>0$, $\delta<\gamma+n$,

$0\leqslant\beta\leqslant\gamma$, $1\leqslant\alpha\leqslant\delta$, $R(\mu-\rho+1+n\,\beta_h)>0$, $(h=1,2,\dots\alpha)$, $a_j-b_h\neq 1,2,3,\dots$, and no two b_j differ by integer

Using the above correspondences in (2), we obtain

$$2\int_0^{\pi/2}\frac{(\cos\theta)^{1-2\rho}(\sin\theta)^{2\mu-1}}{(p+a\cos^2\theta+b\sin^2\theta)^{1+\mu-\rho}}\;G^{\alpha,\beta+n}_{\gamma+n,\delta}\left(\frac{n^n\cos^{2n}\theta}{(p+a\cos\theta+b\sin\theta)^n}\left|\begin{matrix}\wedge(n\,;\rho-\mu),a_1,\dots a_r\\ b_1,\dots,b_\delta\end{matrix}\right.\right)d\theta$$

$$=\frac{\Gamma(\mu)\,(p+a)^{\rho-1}}{n^\mu(p+b)^\mu}\;G^{\alpha,\beta+n}_{\gamma+n,\delta}\left(\frac{n^n}{(p+a)^n}\left|\begin{matrix}\Delta(n\,;\rho),a_1,\dots,a_r\\ b_1,\dots,b_\delta\end{matrix}\right.\right), \tag{3}$$

valid for $\alpha+\beta>\frac{1}{2}(\gamma+\delta-n)$, $R(\mu)>0$, $R(1-\rho)>0$, $R(p)>0$, $R(a)>0$ and $R(b)>0$.

Particular Case: On taking $\alpha=1$, $\beta=\gamma=p$, $\delta=q+1$ and $b_1=0$, we arrive at a result recently obtained by Saxena [4].

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REFERENCES

1. Erdelyi, A. Tables of Integral transforms, Vol. I, *Mc-Graw Hill, New York*. (1954).
2. Hari Shanker, *Journal London Math. Soc*, 23 : 44-49, (1948).
3. Saxena, R. K., Some theorems on generalised Laplace transforms—I, *Proc Nat. Inst. Sci India*, Vol. 26 A, (4) : 400-13, (1960).
4. Saxena, R. K. *Thesis on "A study of Integral transforms" approved for Ph. D. degree of Raj University* (1962).

SOME THEOREMS IN OPERATIONAL CALCULUS—II

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ABSTRACT

In this paper, we have proved three theorems on Laplace transform. By the application of these theorems we have evaluated some infinite integrals involving Bessel, Legendre, and Hypergeometric functions.

1. INTRODUCTION

The Laplace transform is given by

$$\psi(p) = p \int_0^\infty e^{-pt} f(t)\, dt \tag{1}$$

and is symbolically denoted by

$$\psi(p) \doteq f(t) \tag{2}$$

Parseval's formula is :

If $\phi(p) \doteq f(t)$ and $\psi(p) \doteq g(t)$,

then
$$\int_0^\infty \phi(t)\, g(t)\, t^{-1}\, dt = \int_0^\infty \psi(t)\, f(t)\, t^{-1}\, dt \tag{3}$$

The object of the present note is to evaluate some infinite integrals involving Bessel, Legendre and Hypergeometric functions with the help of three theorems in Laplace transform, and generalise the results recently given by the author [5] and Rathie [6]

Theorem I.

If $\psi(p) \doteq f(t)$

and
$$\phi(p) \doteq t^{-\lambda-2\alpha+\beta}\, I_\beta(a/t)\, f(t),$$
then

$$\phi(p) = \frac{p a^\beta 2^{-\beta}}{\Gamma(1+\beta)\,\Gamma(\lambda+2\alpha)} \int_0^\infty t^{\lambda+2\alpha-1}\,(t+p)^{-1}\, {}_0F_3\left(; 1+\beta, \tfrac{1}{2}\lambda+\alpha, \tfrac{1}{2}+\tfrac{1}{2}\lambda+\alpha ; \frac{a^2 t^2}{16}\right) \times \psi(t+p)\, dt \tag{4}$$

provided that the integral is convergent, $R(p) > 0$.

Proof :

Since $f(t) \doteq \psi(p)$

$$e^{-bt} f(t) \doteq \frac{p\psi(p+b)}{(p+b)}, \; R(p) > 0,\; R(b) > 0, \tag{5}$$

and [2, p 220 (19)]

$$\frac{t^{\lambda+2\alpha-1} a^{\beta} 2^{-\beta}}{\Gamma(1+\beta)\Gamma(\lambda+2\alpha)} \ {}_0F_3\left(; 1+\beta, \tfrac{1}{2}\lambda+\alpha, \tfrac{1}{2}+\tfrac{1}{2}\lambda+\alpha ; \frac{a^2t^2}{16}\right)$$

$$\doteqdot p^{1-\lambda-2\alpha+\beta} I_{\beta}(a/p), \ R(\lambda+2\alpha) > 0, R(p) > 0. \tag{6}$$

Using the relations (5) and (6) in (3) we get (4) after replacing b by p.

Example

Take [2, p.146 (29)]

$$f(t) = t^{\lambda+2\alpha-\beta-1} e^{-b/t}$$

$$\doteqdot 2p\left(\frac{b}{p}\right)^{\frac{\lambda+2\alpha-\beta}{2}} K_{\lambda+2\alpha-\beta}(2b^{\frac{1}{2}}p^{\frac{1}{2}})$$

$$= \psi(p), R(p) > 0, R(b) > 0.$$

We then have [2, p. 200 (4)]

$$t^{-\lambda-2\alpha+\beta} I_{\beta}(a/t) f(t) = t^{-1} e^{-b/t} I_{\beta}(a/t)$$

$$\doteqdot 2p \, K_{\beta}\{\sqrt{(b+a)p} + \sqrt{(b-a)p}\} \, I_{\beta}\{\sqrt{(b+a)p} - \sqrt{(b-a)p}\}$$

$$= \phi(p), R(p) > 0, R(b) > | R(a) | .$$

Applying the theorem we get

$$\int_0^{\infty} t^{\lambda+2\alpha-1} (t+p)^{-\frac{(\lambda+2\alpha-\beta)}{2}} K_{\lambda+2\alpha-\beta}\left(2\sqrt{b(t+p)}\right)$$
$${}_0F_3\left(; 1+\beta, \tfrac{1}{2}\lambda+\alpha, \tfrac{1}{2}+\tfrac{1}{2}\lambda+\alpha ; \frac{a^2t^2}{16}\right) dt$$

$$= \frac{\Gamma(1+\beta)\Gamma(\lambda+2\alpha)}{a^{\beta} 2^{-\beta} b^{\frac{\lambda+2\alpha-\beta}{2}}} K_{\beta}\{\sqrt{(b+a)p} + \sqrt{(b-a)p}\} \, I_{\beta}\{\sqrt{(b+a)p} - \sqrt{(b-a)p}\} \tag{7}$$

for $R(p) > 0$, $R(b-a) > 0$, $R(\lambda+2\alpha) > 0$.

If we put $\alpha = \frac{1}{2}\beta$ in (7), and $b \to a$ we get a result due to Rathie [6, p. 63].

Theorem II.

If $\qquad \psi(p) \doteqdot f(t)$

and $\qquad \phi(p) \doteqdot t^{-r} e^{\frac{1}{2}at} W_{k,\mu}(at) f(t)$,

then

$$\phi(p) = \frac{a^k p}{\Gamma(r-k)} \int_0^{\infty} t^{(r-k-1)} (t+p)^{-1} {}_2F_1(\tfrac{1}{2}-k+\mu, \tfrac{1}{2}-k-\mu ; r-k ; -t/a) \, \psi(t+p) \, dt, \tag{8}$$

provided that the integral is convergent, $| \arg a | < \pi$, $R(p) > 0$, $R(r-k) > 0$.

Proof: Using the relation (5) and [2, p. 294 (9)]

$$\frac{a^k}{\Gamma(r-k)} t^{(r-k-1)} {}_2F_1\left(\tfrac{1}{2}-k+\mu, \tfrac{1}{2}-k-\mu ; r-k ; -\frac{t}{a}\right)$$

$$\doteqdot p^{1-r} e^{\frac{1}{2}ap} W_{k,\mu}(ap), R(r-k) > 0, | \arg a | < \pi, R(p) > 0,$$

in (3), we get (8) after replacing b by p.

If we put $k = 0$ and use the relation

$$W_{0,\mu}(x) = (x/\pi)^{\frac{1}{2}} K_{\mu}(x/2),$$

the theorem takes the following form

Corollary :

If $\psi(p) \doteqdot f(t)$

and $\phi(p) \doteqdot t^{\frac{1}{2}-\sigma} e^{at} K_{\mu}(at) f(t),$

then

$$\phi(p) = \frac{p\pi^{\frac{1}{2}}}{\Gamma(\sigma)(2a)^{\frac{1}{2}}} \int_0^{\infty} t^{\sigma-1}(t+p)^{-1} {}_2F_1\left(\tfrac{1}{2}+\mu, \tfrac{1}{2}-\mu; \sigma; -\frac{t}{2a}\right)\psi(t+p)\, dt, \quad (9)$$

provided that the integral is convergent, $R(p) > 0$, $R(\sigma) > 0$, $|\arg a| < \pi$.

Example 1.

Take [4, p. 342]

$$f(t) = t^{\sigma-k-1} I_{\nu}(\gamma t)$$

$$\doteqdot \frac{2^{\frac{1}{2}} p(p^2-\gamma^2)^{\frac{1}{4}-\frac{1}{2}(\sigma+\frac{1}{2}k)}}{\pi^{\frac{1}{2}} \gamma^{\frac{1}{2}}} Q^{\sigma-k-\frac{1}{2}}_{\nu-\frac{1}{2}}\left(\frac{p}{\gamma}\right)$$

$$= \psi(p),\ R(\sigma-k+\nu) > 0,\ R(p) > |R(\gamma)|;$$

we then have [7, p. 175 (5)]

$$t^{-\sigma} e^{\frac{1}{2}at} W_{k,\mu}(at) f(t) = t^{-k-1} e^{\frac{1}{2}at} W_{k,\mu}(at) I_{\nu}(\gamma t)$$

$$\doteqdot \sum_{\mu,-\mu} \frac{a^{\mu+\frac{1}{2}} p\Gamma(-2\mu)\gamma^{\nu}\Gamma(\frac{1}{2}-k+\mu+\nu)}{\Gamma(1+\nu)\Gamma(\frac{1}{2}-\mu-k)2^{\nu}\{\sqrt{p+r}+\sqrt{p-r}\}^{1+2\nu+2\mu-2k}}$$

$$\times F_4\left(\tfrac{1}{2}-k+\mu, \tfrac{1}{2}-k+\mu+\nu; 1+2\mu, 1+\nu; \frac{2a}{p+\sqrt{p^2-\gamma^2}}, \frac{p-\sqrt{p^2-\gamma^2}}{p+\sqrt{p^2-\gamma^2}}\right)$$

$$= \phi(p),\ R(p) > |R(\gamma)|,\ R(\tfrac{1}{2}-k\pm\mu+\nu) > 0.$$

Applying the theorem and replacing p by b^2+c^2, γ by $2bc$, $\frac{1}{2}-k+\mu$ by α, $\frac{1}{2}-k-\mu$ by β, and σ by $\sigma+k+1$, we finally get

$$\int_0^{\infty} t^{\sigma}\{(t+b^2+c^2)^2-4b^2c^2\}^{-\frac{1}{2}(\sigma+\frac{1}{2})} Q^{\sigma+\frac{1}{2}}_{\nu-\frac{1}{2}}\left(\frac{t+b^2+c^2}{2bc}\right)$$

$$\times {}_2F_1\left(\alpha, \beta; \sigma+1; -\frac{t}{a}\right) dt$$

$$= \frac{\pi^{\frac{1}{2}}(bc)^{\nu+\frac{1}{2}}\Gamma(\sigma+1)}{2^{\nu}\Gamma(1+\nu)} \sum_{\alpha,\beta} \frac{\Gamma(\beta-\alpha)\Gamma(\alpha+\nu)(a/c^2)^{\alpha+\nu}}{\Gamma(k)}$$

$$\times F_4\left(\alpha, \alpha+\nu; \alpha-\beta+1, 1+\nu; \frac{a}{c^2}, \frac{b^2}{c^2}\right), \quad (10)$$

for $R(\alpha+1) > 0$, $R(\alpha+\nu) > 0$, $R(\beta+\nu) > 0$, $|\arg a| < \pi$, $R(b \pm c) > 0$.

If we put $b = 0$ in the above we get a known result [3, p. 400].

Example 2.

Take [2, p. 198 (27)]

$$f(t) = t^{\rho-1} e^{-at} K_{\nu}(t)$$

$$\doteqdot \pi^{\frac{1}{2}} \Gamma(\rho+\nu) \Gamma(\rho-\nu) 2^{-\frac{1}{2}} p \{(p+a)^2-1\}^{\frac{1}{4}-\frac{1}{2}\rho} P^{\frac{1}{2}-\rho}_{\nu-\frac{1}{2}}(p+a)$$

$$= \psi(p),\ R(\rho \pm \nu) > 0,\ R(p+a+1) > 0;$$

we then have [8, p. 111]

$$t^{1-\sigma} e^{at} K_{\mu}(at) f(t) = t^{\rho-\sigma} K_{\nu}(t) K_{\mu}(at)$$

$$\doteqdot \sum_{\nu,-\nu} \sum_{\mu,-\mu} \frac{\Gamma(-\mu)\Gamma(-\nu)\, a^{\mu}\, \Gamma(\rho-\sigma+\frac{1}{2}+\mu+\nu)}{2^{\mu+\nu+2}\, p^{\rho-\sigma-\frac{1}{2}+\mu+\nu}}$$

$$\times F_4\left(\frac{\rho-\sigma+\frac{1}{2}+\mu+\nu}{2}, \frac{\rho-\sigma+\frac{3}{2}+\mu+\nu}{2}; 1+\nu, 1+\mu; \frac{1}{p^2}, \frac{a^2}{p^2}\right)$$

$$= \phi(p),\ R(\tfrac{1}{2}+\rho-\sigma \pm \mu \pm \nu) > 0,\ R(p+a+1) > 0.$$

Applying the corollary we get

$$\int_0^{\infty} t^{\sigma-1} \{(t+a+p)^2-1\}^{\frac{1}{4}-\frac{1}{2}\rho} P^{\frac{1}{2}-\rho}_{\nu-\frac{1}{2}}(t+a+p)\, {}_2F_1\left(\tfrac{1}{2}+\mu, \tfrac{1}{2}-\mu; \sigma; -\frac{t}{2a}\right) dt$$

$$= \frac{\Gamma(\sigma)\, \pi^{-1}}{\Gamma(\rho-\nu)\Gamma(\rho+\nu)} \sum_{\nu,-\nu} \sum_{\mu,-\mu} \frac{\Gamma(-\nu)\Gamma(-\mu)\Gamma(\frac{1}{2}+\rho-\sigma+\mu+\nu)\, a^{\mu+\frac{1}{2}}}{2^{\mu+\nu+1}\, p^{\rho-\sigma+\mu+\nu+\frac{1}{2}}}$$

$$\times F_4\left(\frac{\rho-\sigma+\frac{1}{2}+\mu+\nu}{2}, \frac{\rho-\sigma+\frac{3}{2}+\mu+\nu}{2}; 1+\nu, 1+\mu; \frac{1}{p^2}, \frac{a^2}{p^2}\right), \qquad (11)$$

for $R(\frac{1}{2}+\rho-\sigma \pm \mu \pm \nu) > 0$, $R(\sigma) > 0$, $|\arg a| < \pi$, $R(p+a+1) > 0$.

Theorem 3.

If
$$\psi(p) \doteqdot f(t)$$
and
$$\phi(p) \doteqdot t^{\rho-2} e^{-a/2t} W_{k,\mu}(a/t)\, \psi(t).$$
then

$$\phi(p) = p \int_0^{\infty} (t+p)^{-\rho}\, G^{3,0}_{1,3}\left(a(t+p) \,\middle|\, \begin{matrix} 1-k \\ \rho, \frac{1}{2}+\mu, \frac{1}{2}-\mu \end{matrix}\right) f(t)\, dt, \qquad (12)$$

provided that the integral is convergent, $R(p) > 0$, $R(a) > 0$.

Proof:

Since
$$f(t) \doteqdot \psi(p)$$
and [3, p. 412 (54)]

$$t^{\rho-1} e^{-bt} e^{-a/2t} W_{k,\mu}(a/t)$$

$$\doteqdot p(p+b)^{-\rho}\, G^{3,0}_{1,3}\left(a(p+b) \,\middle|\, \begin{matrix} 1-k \\ \rho, \frac{1}{2}+\mu, \frac{1}{2}-\mu \end{matrix}\right),\ R(p+b) > 0,\ R(a) > 0.$$

Using these relations in (3) we get (12) after replacing b by p.

Example

Take [2, p. 278 (23)]

$$f(t) = 2^{-\frac{1}{2}} \pi^{\frac{1}{2}} b^{-\frac{1}{2}} (t^2 + 2bt)^{-\frac{1}{2}\sigma - \frac{1}{4}} \; P_{\nu - \frac{1}{2}}^{\sigma + \frac{1}{2}} (1 + t/b)$$

$$\doteq p^{\sigma + 1} e^{bp} K_\nu (bp)$$

$$= \psi (p),\ R(\sigma) < \tfrac{1}{2},\ |\arg b| < \pi,\ R(p) > 0 ;$$

we then have [9, p. 304]

$$t^{\rho-2} e^{-a/2t} W_{k,\mu} (a/t)\, \psi (t) = t^{\rho+\sigma-1} e^{bt} K_\nu (bt)\, e^{-a/2t} W_{k,\mu} (a/t)$$

$$\doteq \sum_{\nu,-\nu} \sum_{r=0}^{\infty} \frac{b^{\nu+2r}\, \Gamma(-\nu)\, \Gamma(1+\nu)\, \pi^{\frac{1}{2}}}{(r)!\, \Gamma(1+\nu+r)\, 2^{1+\nu+2r}\, (p-b)^{\rho+\nu+\sigma+2r}}$$

$$\times\ G_{1,3}^{3,0}\left(a(p-b) \,\middle|\, \begin{matrix} 1-k \\ \rho+\nu+\sigma+2r,\ \frac{1}{2}+\mu,\ \frac{1}{2}-\mu \end{matrix}\right)$$

$$= \phi (p),\ R(a) > 0,\ R(p) > 0.$$

Applying the theorem we get

$$\int_0^\infty (t+p)^{-\rho} (t^2 + 2bt)^{-\frac{1}{2}\sigma-\frac{1}{4}}\ P_{\nu-\frac{1}{2}}^{\sigma+\frac{1}{2}} (1+t/b)\ G_{1,3}^{3,0}\left(a(t+p) \,\middle|\, \begin{matrix} 1-k \\ \rho,\ \frac{1}{2}+\mu,\ \frac{1}{2}-\mu \end{matrix}\right) dt$$

$$= \sum_{\nu,-\nu} \sum_{r=0}^{\infty} \frac{b^{\frac{1}{2}+\nu+2r}\, \Gamma(-\nu)\, \Gamma(1+\nu)}{(r)!\, \Gamma(1+\nu+r)\, 2^{\frac{1}{2}+\nu+2r}\, (p-b)^{\rho+\nu+\sigma+2r}}$$

$$G_{1,0}^{3,0}\left(a(p-b) \,\middle|\, \begin{matrix} 1-k \\ \rho+\nu+\sigma+2r,\ \frac{1}{2}+\mu,\ \frac{1}{2}-\mu \end{matrix}\right), \tag{13}$$

for $R(a) > 0$, $R(a) > 0$, $R(\sigma) < \frac{1}{2}$, $R(\rho+\sigma \pm \nu) > 0$, $|\arg b| < \pi$.

We shall now evaluate an integral involving products of Bessel function and Hypergeometric function with the help of a theorem earlier given by the author [5]

If $\qquad \psi (p) \doteq f(t)$

and $\qquad \phi (p) \doteq J_\nu (at)\, \psi (1/t)$,

then

$$\phi (p) = 2p \int_0^\infty K_\nu \left\{ \sqrt{2t\,(\sqrt{p^2+a^2}+p)} \right\} J_\nu \left\{ \sqrt{2t(\sqrt{p^2+a^2}-p)} \right\} f(t)\, dt,$$

provided that the integral is convergent, $a > 0$, $R(p) > 0$.

Example

Take [2, p. 220 (19)]

$$f(t) = \frac{t^{\lambda+2\alpha-1}\, b^\beta}{2^\beta\, \Gamma(1+\beta)\, \Gamma(\lambda+2\alpha)}\ {}_0F_3\left(;\ 1+\beta,\ \tfrac{1}{2}\lambda+\alpha,\ \tfrac{1}{2}+\tfrac{1}{2}\lambda+\alpha;\ -\frac{b^2 t^2}{16}\right)$$

$$\doteq p^{1-\lambda-2\alpha+\beta}\, J_\beta (b/p)$$

$$= \psi (p)\ R(\lambda+2\alpha) > 0,\ R(p) > 0 ;$$

then [1, p. 238]

$$J_\nu(at)\,\psi(1/t) = t^{\lambda+2\alpha-\beta-1} J_\nu(at)\, J_\beta(bt)$$

$$\therefore \quad \frac{a^\nu\, b^\beta\, \Gamma(\lambda+\nu+2\alpha)}{2^{\nu+\beta}\,\Gamma(1+\nu)\,\Gamma(1+\beta)\, p^{\lambda+\nu+\beta+2\alpha}}$$

$$\times F_4\left(\frac{\lambda+\nu+2\alpha}{2}, \frac{\lambda+\nu+2\alpha+1}{2} ; 1+\nu, 1+\beta ; -\frac{a^2}{p^2}, -\frac{b^2}{p^2}\right)$$

$$= \phi(p),\ R(p) > 0,\ a > 0,\ b > 0.$$

Applying the theorem and replacing $\sqrt{p^2+a^2}+p$ by $\frac{1}{2}\delta^2$, and $\sqrt{p^2+a^2}-p$ by $\frac{1}{2}\gamma^2$ we get

$$\int_0^\infty t^{\lambda+2\alpha-1} J_\nu\left(\gamma t^{\frac{1}{2}}\right) K_\nu\left(\delta t^{\frac{1}{2}}\right) {}_0F_3\left(; 1+\beta, \tfrac{1}{2}\lambda+\alpha, \tfrac{1}{2}+\tfrac{1}{2}\lambda+\alpha ; -\frac{b^2 t^2}{16}\right) dt$$

$$= \frac{2^{2\lambda+4\alpha-1}(\gamma\delta)^\nu\, \Gamma(\lambda+2\alpha)\,\Gamma(\lambda+\nu+2\alpha)}{\Gamma(1+\nu)\,(\delta^2-\gamma^2)^{\lambda+\nu+2\alpha}}$$

$$\times F_4\left(\frac{\lambda+\nu+2\alpha}{2}, \frac{\lambda+\nu+2\alpha+1}{2} ; 1+\nu, 1+\beta ; -\frac{4\gamma^2\delta^2}{(\delta^2-\gamma^2)^2}, -\frac{16b^2}{(\delta^2-\gamma^2)^2}\right), \qquad (14)$$

for $R(\lambda+\nu+2\alpha) > 0$, $R(\lambda+2\alpha) > 0$, $R(\delta) > \sqrt{2b} + |\,Im\,(\gamma)\,|$.

If we put $\alpha = \beta$ in the above we get a result due to author [5]

REFERENCES

1. Bailley, W. N. Infinite integrals involving Bessel function. *Proc. Lond. Math. Soc.*, 40 : 37-48, (1936).
2. Frdelyi, A. Tables of integral transform Vol. I, *Mc.Graw Hill, New York*, (1954).
3. Erdelyi, A. Tables of integral transforms, Vol. II, *Mc Graw Hill, New York*, (1954).
4. Mac Robert, T. M. Spherical Harmonics, Methuen (1947).
5. Maloo, H. B. Some theorems in Operational Calculus *Proc. Nat. Acad. Sc. India*, accepted for publication.
6. Rathie, C. B. Some infinite integrals involving Bessel functions, *Proc. Nat. Inst. Sc. India*, 20 : 62-69, (1954).
7. Saxena, R. K. Integrals involving Bessel functions and Whittaker functions, *Proc. Camb. Phill. Soc.* 60 : 174-176, (1964).
8. Sharma, K. C. Theorems relating Hankel and Meijer's Bessel transforms. *Proc. Glass Math. Assos.* 6 : 107-112, (1963).
9. Sharma, K. C. Theorems on Meijer's Bessel function transform, *Proc. Nat. Inst. Sc. India*, 30 : 360-366, (1964).

DEFINITE INTEGRALS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTIONS

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ABSTRACT

In this paper a few definite integrals involving generalized hypergeometric functions have been evaluated with the help of the product theorem as is used in operational calculus. The results obtained are supposed to be new.

INTRODUCTION

1. The object of this paper is to evaluate some integrals involving generalized hypergeometric functions with the help of the product theorem of operational calculus, *viz.*

If
$$f_1(p) \doteqdot h_1(x) \text{ and } f_2(p) \doteqdot h_2(x), \text{ then} \tag{1·1}$$

$$\frac{f_1(p)\, f_2(p)}{p} \doteqdot \int_0^x h_1(x-t)\, h_2(t)\,dt$$

$$= x \int_0^{\pi/2} h_1(x\cos^2\theta)\, h_2(x\sin^2\theta)\sin 2\theta\, d\theta \tag{1·2}$$

Thus, if
$$x\, h_1(\lambda x)\, h_2(\mu x) \doteqdot F(p,\lambda,\mu), \text{ then}$$

$$\frac{f_1(p)\, f_2(p)}{p} = \int_0^{\pi/2} F(p, \cos^2\theta, \sin^2\theta)\sin 2\theta\, d\theta, \tag{1·3}$$

provided that the integrals converge.

2. Using the relations (1, pp. 186, 187)

$$f_1(p) = \frac{\Gamma(\rho+\mu)}{\Gamma(2\mu+1)} \frac{\alpha^\mu}{p^{\mu+\rho-1}}\, {}_1F_1\left(\mu+\rho\,;\, 2\mu+1\,;\, -\frac{\alpha}{p}\right)$$

$$\doteqdot h_1(x) = x^{\rho-1} J_{2\mu}(2\alpha^{\frac{1}{2}} x^{\frac{1}{2}}),\ (R(p) > 0,\ R(\rho+\mu) > 0),$$

$$f_2(p) = \frac{\Gamma(\sigma+\nu)}{\Gamma(2\nu+1)} \frac{\beta^\nu}{p^{\sigma+\nu-1}}\, {}_1F_1\left(\sigma+\nu\,;\, 2\nu+1\,;\, -\frac{\beta}{p}\right)$$

$$\doteqdot h_2(x) = x^{\sigma-1} J_{2\nu}(2\beta^{\frac{1}{2}} x^{\frac{1}{2}}),\ (R(p) > 0,\ R(\sigma+\nu) > 0),$$

and

$$x^{\rho-1} J_{2\mu}(2\alpha^{\frac{1}{2}} x^{\frac{1}{2}})\, J_{2\nu}(2\beta^{\frac{1}{2}} x^{\frac{1}{2}})$$

$$\doteqdot \frac{\Gamma(\rho+\nu+\mu)\,\alpha^\mu}{\Gamma(2\mu+1)\,\Gamma(2\nu+1)}\beta^\nu\, p^{1-\rho-\nu-\mu}\, \psi_2\left(\rho+\nu+\mu\,;\, 2\mu+1,\, 2\nu+1\,;\, \frac{\alpha}{p},\, \frac{\beta}{p}\right),$$

$$(R(p) > 0,\ R(\rho+\nu+\mu) > 0),$$

we have, from (1·3)

$$\int_0^{\pi/2} \cos^{2\rho-1}\theta \sin^{2\sigma-1}\theta\, \psi_2\left(\rho+\sigma\,;\lambda,\mu\,;\ \frac{\alpha\cos^2\theta}{p},\ \frac{\beta\sin^2\theta}{p}\right) d\theta$$

$$= \tfrac{1}{2}\,(\rho,\sigma)\ {}_1F_1\,(\rho\,;\lambda\,;-\alpha/p)\ {}_1F_1\,(\sigma\,;\mu\,;-\beta/p),$$

where $R(p)>0$, $R(\rho)>0$, $R(\sigma)>0$ and $\psi_2\,(\ldots;z_1,z_2)$ is the confluent hypergeometric series of two variables (1, p. 385)

3. Taking (1, p. 215)

$$f_1(p) = \Gamma(\mu+\rho+\tfrac{1}{2})\,\alpha^{\mu+\frac{1}{2}}\,p(p+\tfrac{1}{2}\alpha)^{-\mu-\rho-\frac{1}{2}}$$

$$\times\ {}_2F_1\left(\mu+\rho+\tfrac{1}{2},\ \mu-k+\tfrac{1}{2}\,;2\mu+1\,:\ \frac{\alpha}{p+\frac{1}{2}\alpha}\right)$$

$$\doteqdot h_1(x) = x^{\rho-1}\,M_{k,\mu}\,(\alpha x),\ \ (R(p) > \tfrac{1}{2}\,|\,R(\alpha)\,|\,,\ R(\mu+\rho) > -\tfrac{1}{2})$$

and

$$f_2(p) = \Gamma(\nu+\sigma+\tfrac{1}{2})\,\beta^{\nu+\frac{1}{2}}\,p(p+\tfrac{1}{2}\beta)^{-\nu-\sigma-\frac{1}{2}}$$

$$\times\ {}_2F_1\left(\nu+\sigma+\tfrac{1}{2},\ \nu-\lambda+\tfrac{1}{2}\,;2\nu+\,;\ \frac{\beta}{p+\frac{1}{2}\beta}\right)$$

$$\doteqdot h_2(x) = x^{\sigma-1}\,M_{\lambda,\nu}\,(\beta x),\ (R(p) > \tfrac{1}{2}\,|\,R(\beta)\,|\,,\ R(\nu+\sigma) > -\tfrac{1}{2})$$

and using (1, p. 216)

$$x^{\rho-1}\,M_{k,\mu}\,(\alpha x)\,M_{\lambda,\nu}\,(\beta x)$$

$$\doteqdot \Gamma(\rho+\mu+\nu+1)\,\alpha^{\mu+\frac{1}{2}}\,\beta^{\nu+\frac{1}{2}}\,p(p+\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta)^{-\rho-\mu-\nu-1}$$

$$\times\ F_2\left(\rho+\mu+\nu+1\,;\mu+\tfrac{1}{2}-k,\ \nu+\tfrac{1}{2}-\lambda\,;2\mu+1,\ 2\nu+1\,;\ \frac{\alpha}{p+\alpha/2+\beta/2},\ \frac{\beta}{p+\alpha/2+\beta/2}\right)$$

$$(R(\mu+\nu+\rho) > -1,\ R(p \pm \tfrac{1}{2}\alpha \pm \tfrac{1}{2}\beta) > 0),$$

we obtain

$$\int_0^{\pi/2} \cos^{2\rho-1}\theta \sin^{2\sigma-1}\theta\ (p+\tfrac{1}{2}\alpha\cos^2\theta+\tfrac{1}{2}\beta\sin^2\theta)^{-\rho-\sigma}$$

$$\times\ F_2\left(\rho+\sigma\,;\lambda,\mu\,;\nu,\gamma\,;\ \frac{\alpha\cos^2\theta}{p+(\alpha/2)\cos^2\theta+(\beta/2)\sin^2\theta},\ \frac{\beta\sin^2\theta}{p+(\alpha/2)\cos^2\theta+(\beta/2)\sin^2\theta}\right) d\theta$$

$$= \tfrac{1}{2}B(\rho,\sigma)\ {}_2F_1\left(\rho,\lambda\,:\nu\,;\ \frac{\alpha}{p+\frac{1}{2}\alpha}\right)\ {}_2F_1\left(\sigma,\mu;\gamma\,;\ \frac{\beta}{p+\frac{1}{2}\beta}\right)$$

$$\times\ (p+\tfrac{1}{2}\alpha)^{-\rho}\,(p+\tfrac{1}{2}\beta)^{-\sigma},$$

where $R(\rho)>0$, $R(\sigma)>0$, $R(p\pm\alpha/2\pm\beta/2)>0$,

and $F_2(\alpha\,;\beta,\beta^1\,;\gamma,\gamma^1\,;x,y)$ is the hypergeometric series of two variables.

4. By the same procedure, the operational relations (1, pp. 214, 215)

$$f_1(p) = \frac{(-1)^{n-1}(2n)!\,\pi^{\frac{1}{2}}\lambda^{\frac{1}{2}}}{n!\,2^{2n+1/2}}\,p(p+\lambda)^{-n-1}P^1_{2n+1}(p-\lambda)^{1/2}(p+\lambda)^{-1/2}$$

$$\therefore\ h_1(x) = x^{n-\frac{1}{2}}k_{2n+2}(\lambda x),\ (R(p)>0),$$

$$f_2(p) = \frac{(-1)^{m-1}(2m)!\,\pi^{1/2}\mu^{1/2}}{m!\,2^{2m+1/2}}\,p(p+\mu)^{-m-1}P^1_{2m+1}(p-\mu)^{1/2}(p+\mu)^{-\frac{1}{2}}$$

$$\therefore\ h_2(x) = x^{m-\frac{1}{2}}k_{2m+2}(\mu x),\ (R(p)>0),$$

and

$$x^{\rho-1}k_{2n+2}(\lambda x)\,k_{2m+2}(\mu x)$$

$$\doteqdot (-1)^{m+n}2^2\lambda\mu\,\Gamma(\rho+2)\,p(p+\lambda+\mu)^{-\rho-2}$$

$$\times F_2\left(\rho+2\,;-n,-m\,;2,2\,;\frac{2\lambda}{p+\lambda+\mu},\frac{2\mu}{p+\lambda+\mu}\right),$$

$$(R\,\rho)>-2),$$

with the help of (1·3) lead to the result

$$\int_0^{\pi/2}\cos^{2n+2}\theta\,\sin^{2m+2}\theta\,(p+\lambda\cos^2\theta+\mu\sin^2\theta)^{-m-n-3}$$

$$\times F_2\left(m+n+3\,;-n\,;-m\,;2,2\,;\frac{2\lambda\cos^2\theta}{p+\lambda\cos^2\theta+\mu\sin^2\theta},\frac{2\mu\sin^2\theta}{p+\lambda\cos^2\theta+\mu\sin^2\theta}\right)d\theta$$

$$=\frac{\pi(2m)!\,(2n)!\,2^{-2m-2n-4}}{(m+1,\ !\,(n+1)!\,(m+n+2)!}\,\lambda^{-1/2}\mu^{-1/2}(p+\lambda)^{-n-1}(p+\mu)^{-m-1}$$

$$\times P^1_{2n+1}(p-\lambda)^{1/2}.\,(p+\lambda)^{-1/2}.\,P^1_{2m+1}(p-\mu)^{1/2}(p+\mu)^{-1/2}, \tag{4·1}$$

where $R(p)>0$

5. Now we take (1, p 222)

$$f_1(p) = p^{\alpha}\,G^{m,\,n+1}_{h+1,\,k}\left(\frac{\lambda}{p}\,\middle|\,\begin{matrix}\alpha, a_1,\dots,a_h\\ b_1,\dots,b_k\end{matrix}\right)$$

$$\therefore\ h_1(x) = x^{-\alpha}\,G^{m,n}_{h,k}\left(\lambda x\,\middle|\,\begin{matrix}a_1,\dots,a_h\\ b_1,\dots,b_k\end{matrix}\right),$$

where $h+k<2(m+n)$, $h<k$ (or $h=k$ and $R(p)>1$),

$R(a)<\min Re\,b_j+1$, $|\arg p|<(m+n-h/2-k/2)\pi$.

$1\leqslant j\leqslant m$

and (1, p.143)

$$f_2(p) = \frac{p}{p+\beta}\ \therefore\ h_2(x) = e^{-\beta x},\ (R(p+\beta)>0),$$

We accordingly have, from (1·3)

$$\int_0^{\pi/2} \sin\theta \cos^{1-2\alpha}\theta \,(p+\beta\sin^2\theta)^{\alpha-1}\, G^{m,n+1}_{h+1,k}\left(\frac{\lambda \cos^2\theta}{p+\beta\sin^2\theta}\,\middle|\,\begin{matrix}\alpha-1, a_1, \ldots, a_h\\ b_1, \ldots, b_k\end{matrix}\right) d\theta$$

$$= \tfrac{1}{2}\,\frac{p^{\alpha-1}}{p+\beta}\, G^{m,n+1}_{h+1,k}\left(\frac{\lambda}{p}\,\middle|\,\begin{matrix}\alpha, a_1, \ldots, a_h\\ b_1, \ldots, b_k\end{matrix}\right) \tag{5·1}$$

where $h+k < 2(m+n)$, $h < k$ (or $h = k$ and $R(p) > 1$),

$$R(p+\beta) > 0,\ R(\alpha) < \min Re\, b_j + 1,\ j = 1, \ldots, m,$$

and $|\arg p| < (m + n - \frac{1}{2}h - \frac{1}{2}k)\pi$.

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REFERENCES

1. Erdelyi, A. Tables of integral Transforms, Vol I, (1954).
2. Erdelyi, A. Tables of integral Transforms, Vol. II (1954).

ON REPEATED LIMITS AND REPEATED PARTIAL DERIVATIVES

By

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ABSTRACT

We have defined the concept of uniform partial derivatives and strongly uniform derivatives of $f/D \subseteq R^2$. We then generalise the classical theorems of Young and Schwartz on the equality of Repeated partial derivatives.

INTRODUCTION

In the case of a real function f of two variables defined over a domain in R^2 the classical results of Young and Schwartz for $f_{xy}(a, b) = f_{yx}(a, b)$ under certain conditions are well known. We defined uniform partial derivatives, strong uniform derivatives and improved upon the results of Young and Schwartz.[1] Later Hussain[2], also brought some improvement in the same direction. But in all cases the condition was only sufficient. Here we shall introduce a new concept of 'partial uniform limit' and prove a necessary and sufficient condition for the equality of repeated limits (which incidently is also an improvement on the corresponding classical results) and from there derive a necessary and sufficient condition for $f_{xy}(a, b) = f_{yx}(a, b)$.

Definition : Let $V_\eta(a)$ denote the η-neighbourhood of a and $V_\delta(b)$ denote the deleted δ - neighbourhood of b.

If for any $\varepsilon > 0$ and γ there exists a $\delta(\varepsilon)$ and $\eta(\varepsilon, \gamma)$ such that $| f(x,y) - g(x) | < \varepsilon$, provided $x \in V_\eta(a)$ and $y \in V_\delta(b)$, then we define that $\lim_{y \to b} f(x, y) = g(x)$ is partially uniform.

Theorem 1.

Let $f/S \times T$ where S and T are subsets of the real number system R. Let $f(x, y)$ be continuous at $x = a$ for each y and let $\lim_{y \to b} f(x, y)$ to $g(x)$ be continuous at $x = a$. A necessary and sufficient condition for $\lim_{x \to a} \lim_{y \to b} f(x, y)$ and $\lim_{y \to b} \lim_{x \to a} f(x, y)$ to exist and be equal is that $g(x)$ is partially uniform.

Proof of the theorem :

Suppose $g(x)$ is partially uniform.

Then $\lim_{x \to a} \lim_{y \to b} f(x, y) = \lim_{x \to a} g(x) = g(a)$

And $\lim_{y \to b} \lim_{x \to a} f(x, y) = \lim_{y \to b} f(a, y)$

But since $g(x)$ is partially uniform,

$$| f(a, y) - g(a) | < \varepsilon, \text{ provided } y \in V(b)$$

Hence it follows that $\lim_{y \to b} f(a, y) = g(a)$,

Therefore $\lim_{x \to a} \lim_{y \to b} f(x, y) = \lim_{y \to b} \lim_{x \to a} f(x, y)$

Conversely suppose $\lim\limits_{x\to a}\lim\limits_{y\to b} f(x, y) = \lim\limits_{y\to b}\lim\limits_{x\to a} f(x, y)$

Then $\lim\limits_{x\to a}\lim\limits_{y\to b} f(x, y) = \lim\limits_{x\to a} g(x) = g(a)$

Let $\epsilon > 0$ be given.

Now since $\lim\limits_{y\to b} f(x, y) = g(x)$ for each x and in particular at $x = a$, there exists a $\delta(\epsilon)$ such that $| f(a, y) - g(a) | < \epsilon/3$, provided $y \in V_\delta(b)$.

Next since $g(x)$ is continuous at $x = a$, there exists a $\delta_1(\epsilon)$ such that $| g(x) - g(a) | < \epsilon/3$, provided $x \in V\delta_1(a)$. Lastly since $f(x, y)$ is continuous at $x = a$ for each y, for any given η there exists a $\delta_2(\epsilon, \gamma)$ such that $| f(x, y) - f(a, y) | < \epsilon/3$ provided $x \in V\delta_1(a)$.

Now letting $\eta = \min(\delta_1, \delta_2)$, we find that for any $\epsilon > 0$ and there exists a $\eta(\epsilon, \gamma)$ such that $| f(x, y) - g(x) | \, | f(x, y) - f(a, y) | + | f(a, y) - g(a) | + | g(a) - g(x) | < \epsilon$, provided $x \in V_\eta(a)$ and $y \in V_\delta(b)$.

This means that $g(x)$ is partially uniform.

There is a companion theorem to theorem (1) in the following form :

Theorem 2.

Let $f/S\,X\,T$. Let $f(x, y)$ be continuous at $y = b$ for each x and let $\lim\limits_{x\to a} f(x, y)$ to $h(y)$ be continuous at $y = b$. A necessary and sufficient condition for $\lim\limits_{x\to a}\lim\limits_{y\to b} f(x, y)$ and $\lim\limits_{y\to b}\lim\limits_{x\to a} f(x, y)$ to exist and be equal is that $h(y)$ is partially uniform.

The proof follows on the same lines as given in theorem (1). Next we consider the question of $f_{xy}(a, b) = f_{yx}(a, b)$ for a function $f/S\,X\,T$.

We now suppose that in a certain neighbourhood $V(a, b)$, f_x and f_y both exist. We consider a function

$$\frac{\Delta(h, k)}{h\,k} = \frac{f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)}{h\,k}$$

where $h \neq 0$, $k \neq 0$.

Obviously the single limits $\lim\limits_{h\to 0} \frac{\Delta(h, k)}{h\,k}$ and $\lim\limits_{k\to 0} \frac{\Delta(h, k)}{h\,k}$ exist. We then define $\frac{\Delta(h, k)}{h\,k}$ elsewhere suitably so that $\frac{\Delta(h, k)}{h\,k}$ is continuous at $h = 0$, for all k, and $\lim\limits_{k\to 0} \frac{\Delta(h, k)}{h\,k}$ is continuous at $h = 0$, then theorem (1) applies and we get the following result.

Theorem 3.

A necessary and sufficient condition for $f_{xy}(a, b)$ and $f_{yx}(a, b)$ to exist and be equal is that the $\lim\limits_{h\to 0} \frac{\Delta(h, k)}{h\,k}$ is partially uniform.

REFERENCES

1. Shukla, R. 'On uniform derivatives'. Bihar University Journ. 1957, Science Number.
2. Hussain, Z. Ph. D. Thesis, Bihar University, 1962.

DETERMINATION OF THE CURVATURE OF A MAGNETIC LINE BEHIND A THREE DIMENSIONAL UNSTEADY CURVED SHOCK WAVE

By

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ABSTRACT

In this paper the curvature k of a magnetic line just behind a three dimensional unsteady curved shock wave has been calculated in principle. These are the extension of our previous results [1] which have been calculated explicitly for two dimensional flow of a conducting gas. A coordinate system with magnetic line as one of the coordinate curves is employed. Furthermore it is assumed that the magnetic field H^i and the velocity field V^i are codirectional on both sides of the shock surface.

1. INTRODUCTION

Let the shock configuration in three dimensional unsteady motion of a gas be given by

$$x^i = x^i(y^1, y^2, t)^{2)},$$

where x^i are the rectangular Cartesian coordinates of a point P and y^1, y^2, are the Gaussian coordinates of P on the shock surface.

Then the first fundamental tensor of the shock surface is given by

$$(1\cdot1) \qquad a_{\alpha\beta} = \partial_\alpha x^i \, \partial_\beta x^i,^{3)} \text{ where } \partial_\alpha x^i = \frac{\partial x^i}{\partial y^\alpha}$$

As the shock surface is real, det $\| a_{\alpha\beta} \|$ is of rank 2 and there exists an inverse tensor $a^{\alpha\beta}$ of $a_{\alpha\beta}$ such that

$$(1\cdot2) \qquad a_{\alpha\gamma} \, a^{\alpha\beta} = \delta^\beta_\gamma ,$$

where δ^β_γ are the kronecker deltas, having the values $\begin{cases} 1 \\ 0 \end{cases}$ according as $\begin{cases} \beta = \gamma \\ \beta \neq \gamma \end{cases}$.

The tensors $a_{\alpha\beta}$ or $a^{\alpha\beta}$ will be used to lower or raise the Greek indices. For simplicity we take lines of curvature as the Gaussian coordinate curves on the

1) Numbers in brackets refer to the references at the end of the paper.

2) In this and in what follows Latin indices will range from 1 to 3 and the Greek indices will take the values 1, 2.

3) A repeated index implies summation.

shock surface. The vector whose components are $\partial_\alpha x^j$ is tangential to the shock surface. The unit space vectors tangent to the coordinate curves are

$$(1\cdot3) \qquad \frac{1}{\sqrt{a_{11}}}\,\partial_1 x^i \quad \text{and} \quad \frac{1}{\sqrt{a_{22}}}\,\partial_2 x^i .$$

The components X^i of the unit vector normal to the shock surface directed from the region infront to the region behind the shock surface are given by

$$(1\cdot4) \qquad X_i = \tfrac{1}{2}\,\epsilon^{\alpha\beta}\,\epsilon_{ijk}\,\partial_\alpha x^j\,\partial_\beta x^k$$

where ϵ_{ijk} is an isotropic tensor having the values 1 or -1 according as ijk is an even or odd permutation of 123 and the value 0 when i, j, k, are not all different. The permutation tensor $\epsilon_{\alpha\beta}$ of the surface has the values

$$\epsilon^{11} = \epsilon^{22} = 0\;;\; \epsilon^{12} = -\epsilon^{21} = \frac{1}{\sqrt{a}},$$

where

$$a \overset{\text{def}}{=} \det\| a_{\alpha\beta} \| \,.$$

The components $b_{\alpha\beta}$ of the second fundamental form of the surface are given by

$$(1\cdot5) \qquad b_{\alpha\beta} = \tfrac{1}{2}\,\epsilon^{\gamma\delta}\,\epsilon_{ijk}\,\partial_\beta\,\partial_\alpha x^i\,\partial_\gamma x^j\,\partial_\delta x^k.$$

Normal curvatures in the direction of coordinate curves are

$$(1\cdot6) \qquad K_\alpha = b_{\alpha\alpha}/a_{\alpha\alpha} \ (\alpha \text{ not summed}).$$

For the derivative of $\partial_\alpha x^i$ with respect to y^β, we have

$$(1\cdot7) \qquad \partial_\beta\,\partial_\alpha x^i - \partial_\gamma x^i \left\{ \begin{matrix} \gamma \\ \alpha\beta \end{matrix} \right\} = b_{\alpha\beta} X^i ,$$

where $\left\{ \begin{matrix} \gamma \\ \alpha\beta \end{matrix} \right\}$ are christoffel symbols with respect to $a_{\alpha\beta}$.

Weingarten's formula for the derivative of the unit normal is

$$(1\cdot8) \qquad \partial_\alpha X^i = -a^{\beta\gamma}\,b_{\beta\alpha}\,\partial_\gamma x^i.$$

As the lines of curvature are the Gaussian coordinate curves, we have

$$a_{12} = b_{12} = a_{21} = b_{21} = 0,$$

and from (1·2) we have

$$a^{12} = a^{21} = 0,\; a^{11} = a^{22}/a,\; a^{22} = a^{11}/a.$$

If viscosity, heat conductivity and electric resistance are absent then the equations governing three dimensional unsteady motion of a continuous conducting gas are[2]

$$(1\cdot9) \qquad \frac{\delta\rho}{\delta t} + V^i\,\partial_i\rho + \rho\partial_i U^i = 0,$$

(1·10) $$\rho \frac{\delta U^i}{\delta t} + \rho V^j \partial_j U^i + \partial_i p + \frac{1}{4\pi} H^k \partial_i H^k - \frac{1}{4\pi} H^j \partial_j H^i = 0,$$

(1·11) $$\frac{\delta H^i}{\delta t} + V^j \partial_j H^i - H^j \partial_j U^i + H^i \partial_k U^k = 0,$$

(1·12) $$\partial_i H^i = 0,$$

where H^i, U^i, p and ρ denote components of magnetic field, velocity components, pressure and density respectively and ∂_i denotes the partial derivative with respect to x^i. The quantities $V^i = U^i - G X^i$ where $G X^i$ are the components of the shock velocity and $\delta/\delta t$ denotes the time derivative as apparent to an observer moving with the velocity of the shock.

A quantity f if evaluated infront of the shock surface will be denoted by $f_{1/}$, if in the region behind the shock surface, then by f_{1}. The jump in f is given by

$$[f] \stackrel{\text{def}}{=} f_l - f_1 .$$

The expressions for flow and field quantities behind the shock separately in terms of their values infront of the shock are[8]

(1·13) $$[H_i] = S_H (H_{1/i} - H_{1n/} X_i),$$

(1·14) $$\Big[U_i \Big] = \frac{V_{1n/}}{H_{1n/}} A_{1/} S_H H_{1/i} - V_{1n/} \frac{(1+A_{1/} S_H)}{1+S_H} S_H X_i,$$

(1·15) $$\Big[p \Big] = S_H \frac{(1-A_{1/})}{1+S_H} \rho_{1/} V^2_{1n/} - \frac{1}{8\pi} S_H (2+S_H) H_{1/\alpha} H^{\alpha}_{1/}$$

(1·16) $$\Big[\rho \Big] = \frac{\rho_{1/} S_H (1-A_{1/})}{1+A_{1/} S_H},$$

where S_H is the magnetic field strength of the shock defined as

(1·17) $$S_H H_{1/\alpha} = [H_\alpha],$$

and

(1·18 *a*) $$A_{1/} = \frac{H^2_{1n/}}{4\pi\rho_{1/} V^2_{1n/}},$$

(1·18 *b*) $$H_{1n/} = H_{1/i} X^i,$$

(1·18 *c*) $$U_{1n/} = U_{1/i} X^i,$$

(1·18 *d*) $$V_{1n/} = V_{1/i} X^i,$$

(1·18 *e*) $$H_{1/\alpha} = H\beta_{1/} a_{\alpha\beta} = H_{1/i} \partial_\alpha x^i.$$

(1·18 *f*) $$H_{1n/} = H_{n/} .$$

For a perfect gas, S_H is given by the relation

(1·19) $$C^2_{1/}(A_{1/}-1) = \frac{(A_{1/}-1)\, V^2_{1n/}}{2(1+S_H)} \left(2+A_{1/}\, S_H+\gamma A_{1/}\, S_H-\gamma S_H+S_H \right)$$

$$+ \frac{A_{1/}\, V^2_{1n/}}{2H^2_{1n/}}\, H_{1/\alpha}\, H^{\alpha}_{1/} \left(2+A_{1/}\, S_H+2S_H+A_{1/}\, S^2_H+\gamma A_{1/}\, S_H-\gamma S_H \right),$$

where the velocity of souud $C_{1/}$ is given by

(1·20) $$C^2_{1/} = \gamma p_{1/}/\rho_{1/},$$

γ being the ratio of two specific heats C_p and C_v, assumed constant.

2. ANOTHER COORDINATE SYSTEM

At any point x^i behind the shock surface let ds be the elementary arc length along a magnetic line ; there the components of the unit tangent vector to the magnetic line are given by

(2·1) $$\frac{\partial x^i}{\partial s} = \frac{H^i}{H} \; ; \; H^2 \equiv H_i\, H^i.$$

Consider a surface S through a point x^i behind the shock surface at a distance s from the shock surface along a magnetic line, which coincides with the shock surface $\underset{o}{S}$ when s tends to zero. The equation of this surface is

(2·2) $$x^i = x^i(y^1, y^2, s),$$

with the initial conditions

$$x^i(y^1, y^2, 0) = \underset{o}{x}^i(y^1\, y^2),$$

and

$$X^i(y^1, y^2, 0) = \underset{o}{X}^i(y^1, y^2)^{4)}.$$

Since X^i and $\partial_\alpha x^i$ are three noncoplanar vectors we can express V^i and H^i in terms of these by the relations

(2·3) $$V^i = V_{n/}\, X^i + V^\alpha\, \partial_\alpha\, x^i,$$

(2·4) $$H^i = H_{n/}\, X^i + H^\alpha\, \partial_\alpha\, x^i,$$

where

(2·5 *a*) $$V_\alpha = V^\beta\, a_{\alpha\beta} = V_i\, \partial_\alpha\, x^i, \; ; \; H_\alpha = H_i\, \partial_\alpha\, x^i,$$

and

(2·5 *b*) $$V_{n/} = V_i\, X^i \; ; \; H_{n/} = H_i\, x^i.$$

At any point Q behind the shock surface, the flow variables are functions of y^1, y^2 and s, so in view of the above transformation we have

(2·6) $$\partial_j\, x^i = \delta^i_j = \frac{H^i}{H}\, \partial_j s + \partial_\alpha\, x\, \partial_j\, y^\alpha,$$

and

(2·7) $$\partial_j f = \frac{\partial f}{\partial s}\, \partial_j s + \partial_\alpha f\, \partial_j\, y^\alpha.$$

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4) The kernel letter o below any quantity denote its value at the shock surface.

Now multiplying (2·6) by X_i we obtain

(2·8) $$X^j = \frac{H_{n'}}{H} \partial_j s.$$

Again multiplying (2·6) by $\varepsilon^{\gamma\beta} \varepsilon_{ilk} H^l \partial_\beta x^k$ and using the relations

(2·9) $$\varepsilon_{ilk} H^i H^l = 0,$$

and

(2·10) $$\varepsilon_{\alpha\beta} X_i = \varepsilon_{ijk} \partial_\alpha x^j \partial_\beta x^k,$$

we get

(2·11) $$\partial_j y^\alpha = \frac{1}{H_{n'}} \varepsilon^{\alpha\beta} \varepsilon_{ijk} H^i \partial_\beta x^k.$$

Multiplying this equation by H^j and using (2·9) we get

(2·12) $$H^j \partial_j y^\alpha = 0.$$

Further multiplying (2·11) by X^j and using the relations

(2·13) $$\varepsilon_{ijk} X^j X^k = 0,$$

and

(2·14) $$\varepsilon_{\alpha\beta} = \varepsilon_{ijk} X^i \partial_\alpha x^j \partial_\beta x^k,$$

we obtain

(2·15) $$X^j \partial_j y^\alpha = -\frac{H^\alpha}{H_{n'}}.$$

Furthermore multiplying (2·11) by $\partial_\gamma x^j$ and using (2·10) we obtain

(2·16) $$\partial_\gamma x^j \partial_j y^\alpha = \delta^\alpha_\gamma.$$

Multiplying (2·11) by V^j, using (2·3), (2·15) and (2·16) we obtain

(2·17) $$V^j \partial_j y^\alpha = V^\alpha - V_{n'} \frac{H^\alpha}{H_{n'}}.$$

Further we assume the relations

(2·18) $$\frac{H^\alpha_{1'}}{H_{1n'}} = \frac{V^\alpha_{1'}}{V_{1n'}}; \quad \frac{H^\alpha}{H_{n'}} = \frac{V^\alpha}{V_{n'}},$$

then (2·17) assumes the form

(2·19) $$V^j \partial_j y^\alpha = 0,$$

3. DETERMINATION OF CURVATURE

By virtue of the relations (2·7), (2·8), (2·12) and (2·19) the equations (1·9), (1·10), (1·11), and (1·12) assume respectively the forms

(3·1) $$\frac{\delta \rho}{\delta t} + H \frac{V_{n'}}{H_{n'}} \frac{\partial \rho}{\partial s} + \rho X^i \frac{H}{H_{n'}} \frac{\partial U^i}{\partial s} + \rho \partial_\alpha u^i \partial_i y^\alpha = 0,$$

(3·2) $$\rho \frac{\delta u^i}{\delta t} + H\rho \frac{V_{n_1}}{H_{n_1}} \frac{\partial u^i}{\partial s} - \frac{H}{4\pi} \frac{\partial H^i}{\partial s} + \frac{H}{H_{n_1}} \frac{\partial p}{\partial s} X^i$$
$$+ \frac{1}{4\pi} \frac{H}{H_{n_1}} H^k \frac{\partial H^k}{\partial s} X^i + \left(\partial_\alpha p + \frac{1}{4\pi} H^k \partial_\alpha H^k \right) \partial_i y^\alpha = 0,$$

(3·3) $$\frac{\delta H^i}{\delta t} + H \frac{V_{n_1}}{H_{n_1}} \frac{\partial H^i}{\partial s} - H \frac{\partial u^i}{\partial s} + H^i \frac{H}{H_{n_1}} X^j \frac{\partial u^j}{\partial s} + H^i \partial_\alpha u^j \partial_j y^\alpha = 0,$$

and

(3·4) $$\frac{H}{H_{n_1}} X^i \frac{\partial H^i}{\partial s} + \partial_\alpha H^i \partial_i y^\alpha = 0,$$

Further we assume that the gas satisfies the relation $p = p(\rho)$. Differentiating the above relation with respect to s, we obtain

(3·5) $$\frac{\partial p}{\partial s} = c^2 \frac{\partial \rho}{\partial s} \text{ where } c^2 = \frac{dp}{d\rho}.$$

Now multiplying (3 3) by X^i we obtain

$$X^i \frac{\delta H^i}{\delta t} + H \frac{V_{n_1}}{H_{n_1}} X^i \frac{\partial H^i}{\partial s} H_{n_1} \partial_\alpha u^j \partial_j y^\alpha = 0.$$

Also multiplying (3·4) by V_{n_1} and comparing with the above equation we obtain

(3·6) $$X^j \frac{\delta H^j}{\delta t} + H_{n_1} \partial_\alpha u^j \partial_j y^\alpha = V_{n_1} \partial_\alpha H^j \partial_j y^\alpha.$$

Further substituting the value of $X^j \frac{\partial u^j}{\partial s}$ from (3·1) in the equation (3·3) we obtain

(3·7) $$\frac{\delta H^i}{\delta t} - H \frac{\partial u^i}{\partial s} + H \frac{V_{n_1}}{H_{n_1}} \frac{\partial H^i}{\partial s} - \frac{H}{\rho} \frac{V_{n_1}}{H_{n_1}} \frac{\partial \rho}{\partial s} H^i - \frac{H^i}{\rho} \frac{\delta \rho}{\delta t} = 0.$$

Multiplying (3·2) by H^i and using (3·5) and (2·12) we obtain

(3 8) $$\rho H^i \frac{\delta u^i}{\delta t} + H\rho \frac{V_{n_1}}{H_{n_1}} H^i \frac{\partial u^i}{\partial s} + H c^2 \frac{\partial \rho}{\partial s} = 0.$$

Furthermore multiplying (3·7) by H^i and substituting the value of $H^i \frac{\partial u^i}{\partial s}$ from (3·8) we obtain

(3·9) $$H \frac{V_{n_1}}{H_{n_1}} H^i \frac{\partial H^i}{\partial s} = \frac{\partial \rho}{\partial s} \left(\frac{H^3}{\rho} \frac{V_{n_1}}{H_{n_1}} - \frac{H_{n_1}}{V_{n_1}} \frac{H}{\rho} c^2 \right) - H^i \frac{\delta H^i}{\delta t}$$
$$- \frac{H_{n_1}}{V_{n_1}} H^i \frac{\delta u^i}{\delta t} + \frac{H^2}{\rho} \frac{\delta \rho}{\delta t}.$$

Multiplying (3·2) by X^i we get

(3·10) $$\rho \frac{\delta u^i}{\delta t} X^i + H\rho \frac{V_{n_1}}{H_{n_1}} X^i \frac{\partial u^i}{\partial s} - \frac{H}{4\pi} X^i \frac{\partial H^i}{\partial s} + \frac{H}{H_{n_1}} \frac{\partial p}{\partial s}$$
$$+ \frac{1}{4\pi} \frac{H}{H_{n_1}} H^k \frac{\partial H^k}{\partial s} + \left(\partial_\alpha p + \frac{1}{4n} H^k \partial_\alpha H^k \right) X^i \partial_i y^\alpha = 0.$$

With the help of (3·1), (3·3), (3·4) and (3·5) the above equation assumes the form

$$(3{\cdot}11)\qquad H\frac{V^2_{nl}}{H_{nl}}M\frac{\partial \rho}{\partial s}=\rho V_{nl}\,\partial_\alpha u^i\,\partial_i y^\alpha-\frac{H_{nl}}{4\pi}\partial_\alpha H^i\,\partial_i y^\alpha$$

$$-\left(\partial_\alpha p+\frac{1}{4\pi}H^k\partial_\alpha H^k\right)X^i\partial_i y^\alpha-\rho\frac{\delta u^i}{\delta t}X^i+\frac{H^k}{4\pi V_{nl}}\frac{\delta H^k}{\delta t}$$

$$+\frac{H_{nl}}{4\pi V^2_{nl}}H^k\frac{\delta u^k}{\delta t}+V_{nl}\frac{\delta\rho}{\delta t}\left(1-\frac{H^2}{4\pi\rho V^2_{nl}}\right),$$

where

$$M=\left(1-\frac{c^2}{V^2_{nl}}\right)(A-1)+\frac{H^\alpha H_\alpha}{4\pi\rho V^2_{nl}}$$

In consequence of (3·6), the equation (3·11) assumes the form

$$(3{\cdot}12)\qquad H\frac{V^2_{nl}}{H_{nl}}M\frac{\partial\rho}{\partial s}=\frac{\rho V^2_{nl}}{H_{nl}}(1-A)\,\partial_\alpha H^i\,\partial_i y^\alpha$$

$$-\left(\partial_\alpha p+\frac{1}{4\pi}H^k\partial_\alpha H^k\right)X^i\partial_i y^\alpha$$

$$-\frac{\rho V_{nl}}{H_{nl}}X^i\frac{\delta H^i}{\delta t}-\rho\frac{\delta u^i}{\delta t}X^i+\frac{H^k}{4\pi V_{nl}}\frac{\delta H^k}{\delta t}$$

$$+\frac{H_{nl}}{4\pi V^2_{nl}}H^k\frac{\delta u^k}{\delta t}+\left(V_{nl}-\frac{H^2}{4\pi\rho V_{nl}}\right)\frac{\delta\rho}{\delta t}$$

Furthermore substituting in (3·2) the values of $\frac{\partial u^i}{\partial s}$ and $H^k\frac{\partial H^k}{\partial s}$ from (3·7) and (3·9) we obtain

$$\rho H\frac{V^2_{nl}}{H^2_{nl}}(1-A)\frac{\partial H^i}{\partial s}=-\frac{H}{H_{nl}}V^2_{nl}\left(MX^i-\frac{H^\alpha}{H_{nl}}\partial_\alpha X^i\right)\frac{\partial\rho}{\partial s}$$

$$-\left(\partial_\alpha p+\frac{1}{4\pi}H^k\partial_\alpha H^k\right)\partial_i y^\alpha-\rho\frac{\delta u^i}{\delta t}-\frac{\rho V_{nl}}{H_{nl}}\frac{\delta H^i}{\delta t}$$

$$+\frac{V_{nl}}{H_{nl}}H^i\frac{\delta\rho}{\delta t}+\frac{1}{4\pi}\frac{H^k}{V_{nl}}\frac{\delta H^k}{\delta t}X^i$$

$$+\frac{1}{4\pi}\frac{H_{nl}}{V^2_{nl}}H^k\frac{\delta u^k}{\delta t}X^i-\frac{1}{4\pi\rho V_{nl}}H^2\frac{\delta\rho}{\delta t}X^i$$

Multiplying this equation by M and substituting the value of $\frac{\partial\rho}{\partial s}$ from (3·12) we obtain

$$(3{\cdot}13)\qquad MH\rho\frac{V^2_{nl}}{H^2_{nl}}(1-A)\frac{\partial H^i}{\partial s}=-MX^i\left\{\frac{\rho V^2_{nl}}{H_{nl}}(1-A)\,\partial_\alpha H^j\,\partial_j y^\alpha\right.$$

$$+\frac{H^\alpha}{H_{nl}}\left(\partial_\alpha p+\frac{1}{4\pi}H^k\partial_\alpha H^k\right)-\frac{\rho V_{nl}}{H_{nl}}X^j\frac{\delta H^j}{\delta t}-\rho\frac{\delta u^j}{\delta t}X^j$$

$$+\frac{H^k}{4\pi V_{n_1}}\frac{\delta H^k}{\delta t}+\frac{H_{n_1}}{4\pi V^2_{n_1}}H^k\frac{\delta u^k}{\delta t}+\left(V_{n_1}-\frac{H^2}{4\pi\rho V_{n_1}}\right)\frac{\delta\rho}{\delta t}$$

$$-\frac{1}{4\pi}\frac{H^k}{V_{n_1}}\frac{\delta H^k}{\delta t}-\frac{H_{n_1}}{4\pi V^2_{n_1}}H^k\frac{\delta u^k}{\delta t}+\frac{1}{4\pi\rho V_{n_1}}H^2\frac{\delta\rho}{\delta t}\Big]\Big\}$$

$$+\frac{H^\beta}{H_{n_1}}\partial_\beta X^i\left\{\frac{\rho V^2_{n_1}}{H_{n_1}}(1-A)\,\partial_\alpha H^j\,\partial_j y^\alpha+\frac{H^\alpha}{H_{n_1}}\left(\partial_\alpha p+\frac{1}{4\pi}H^k\partial_\alpha H^k\right)\right.$$

$$-\frac{\rho V_{n_1}}{H_{n_1}}X^j\frac{\delta H^j}{\delta t}-\rho\frac{\delta u^j}{\delta t}X^j+\frac{H^k}{4\pi V_{n_1}}\frac{\delta H^k}{\delta t}+\frac{H_{n_1}}{4\pi V^2_{n_1}}H^k\frac{\delta u^k}{\delta t}$$

$$\left.+\left(V_{n_1}-\frac{H^2}{4\pi\rho V_{n_1}}\right)\frac{\delta\rho}{\delta t}\right\}-M\left\{\left(\partial_\alpha p+\frac{1}{4\pi}H^k\partial_\alpha H^k\right)\partial_i y^\alpha\right.$$

$$\left.+\rho\frac{\delta u^i}{\delta t}+\frac{\rho V_{n_1}}{H_{n_1}}\frac{\delta H^i}{\delta t}-\frac{V_{n_1}}{H_{n_1}}\frac{\delta\rho}{\delta t}H^i\right\}$$

The above relation gives the value of $\frac{\partial H^i}{\partial s}$ at any point Q behind the shock surface. If we make $s\to o$, then (3·13) gives the value of $\frac{\partial H^i}{\partial s}$ just behind the shock surface.

To find the value of $\partial_\alpha H^i$, $\partial_\alpha p$, $\frac{\delta u^i}{\delta t}$, $\frac{\delta\rho}{\delta t}$ and $\frac{\delta H^i}{\delta t}$ we assume that the flow and field ahead of the shock is uniform.

Differentiating the relations (1·13), (1·15) and (1·19) with respect to y^α we obtain

(3·14) $$\partial_\alpha H^i=-S_H\left(X^i\partial_\alpha H_{1n_1}+H_{1n_1}\partial_\alpha X^i\right)+H^\beta_{1_1}\partial_\beta X^i\,\partial_\alpha S_H,$$

(3·15) $$\partial_\alpha p=\partial_\alpha S_H\left(\frac{(1-A_{1_1})}{(1+S_H)^2}\rho_{1_1}V^2_{1n_1}-\frac{1}{4\pi}(1+S_H)\,H_{1_1\delta}H^\delta_{1_1}\right)$$
$$+\frac{S_H}{1+S_H}2\rho_{1_1}V_{1n_1}\partial_\alpha V_{1n_1}+\frac{3+S_H}{1+S_H}S^2_H\frac{1}{4\pi}H_{1n_1}\partial_\alpha H_{1n_1},$$

and

(3·16) $$\partial_\alpha S_H\left\{2c^2_{1_1}(A_{1_1}-1)-(A_{1_1}-1)(A_{1_1}+\gamma A_{1_1}-\gamma+1)V^2_{1n_1}\right.$$
$$\left.-\left(3A_{1_1}S^2_H+(4A_{1_1}+4+2\gamma A_{1_1}-2\gamma)S_H+A_{1_1}+\gamma A_{1_1}-\gamma+4\right)\frac{H_{1_1\delta}H^\delta_{1_1}}{4\pi\rho_{1_1}}\right\}$$
$$=\frac{H_{1n_1}\partial_\alpha H_{1n_1}}{2\pi\rho_{1_1}}\left\{\frac{1}{4\pi\rho_{1_1}V^2_{1n_1}}S_H(1+S_H)(\gamma+1+S_H)\,H_{1_1\delta}H^\delta_{1_1}-\frac{2c^2_{1_1}}{V^2_{1n_1}}(1+S_H)\right.$$
$$\left.-A_{1_1}S^3_H+S^2_H(\gamma-2-2A_{1_1}-\gamma A_{1_1})+S_H(\gamma A_{1_1}+A_{1_1}-4-\gamma)\right\}$$
$$-2V_{1n_1}\partial_\alpha V_{1n_1}\left\{\frac{A_{1_1}}{4\pi\rho_{1_1}V^2_{1n_1}}S_H(1+S_H)(\gamma+1+S_H)H_{1_1\delta}H^\delta_{1_1}\right.$$
$$\left.-\frac{2c_{1_1}^2}{V^2_{1n_1}}A_{1_1}(1+S_H)+S_H(\gamma A^2_{1_1}+A^2_{1_1}+1-\gamma)+2\right\},$$

where

(3·17) $$\partial_\alpha V_{1n_1} = - u_{1,\gamma}\, a^{\beta\gamma}\, b_{\beta\alpha} - \partial_\alpha G,$$

(3·18) $$\partial_\alpha H_{1n_1} = - H_{1,\gamma}\, a^{\beta\gamma}\, b_{\beta\alpha}$$

Again applying the operator $\delta/\delta t$ on the equations (1·13), (1·14) and (1·16) we obtain

(3·19) $$\frac{\delta H_i}{\delta t} = - S_H \left(X^i \frac{\delta H_{1n_1}}{\delta t} + H_{1n_1} \frac{\delta X^i}{\delta t} \right) + H^{\beta}_{1,}\, \partial_\beta\, x^i\, \frac{\delta S_H}{\delta t},$$

(3·20) $$\frac{\delta u^i}{\delta t} = \frac{\delta S_H}{\delta t} \left(\frac{V_{1n_1}}{H_{1n_1}} A_{1,}\, H^{\beta}_{1,}\, \partial_\beta\, x^i - \frac{V_1 n_1}{(1+S_H)^2} (1 - A_{1,}) X^i \right.$$
$$+ \frac{\delta H_{1n_1}}{\delta t} \left(\frac{S_H}{4\pi\rho_{1,} V_{1n_1}} H^{\beta}_{1,}\, \partial_\beta\, x^i + \frac{S_H H_1 n_1}{4\pi\rho_{1,} V_1 n_1} \frac{(1-S_H)}{(1+S_H)} X^i \right)$$
$$+ \frac{\delta V_{1n_1}}{\delta t} \left(- \frac{A_{1,}}{H_{1n_1}} S_H H^{\beta}_{1,}\, \partial_\beta\, x^i - \frac{S_H}{1+S_H} (1 + A_{1,}) X^i \right)$$
$$+ \frac{\delta X^i}{\delta t} \left(- S_H V_{n_1} \right),$$

(3·21) $$\frac{\delta\rho}{\delta t} = \frac{\rho_{1,}(1 - A_{1,})}{(1 + A_{1,}S_H)^2} \frac{\delta S_H}{\delta t} - \frac{2A_{1,} S_H (1 + S_H)\, \rho_{1,}}{(1 + A_{1,} S_H)^2} \left(\frac{1}{H_{1n_1}} \frac{\delta H_{1n_1}}{\delta t} - \frac{1}{V_{1n_1}} \frac{\delta V_{1n_1}}{\delta t} \right),$$

where

(3·22) $$\frac{\delta X^i}{\delta t} = - a^{\alpha\beta}\, \partial_\alpha G\, \partial_\beta\, x^i\ ,$$

(3·23) $$\frac{\delta V_1 n_1}{\delta t} = - \left(u_{1,\beta}\, a^{\alpha\beta}\, \partial_\alpha G + \frac{\delta G}{\delta t} \right),$$

(3·24) $$\frac{\delta H_1 n_1}{\delta t} = - H_{1,\beta}\, a^{\alpha\beta}\, \partial_\alpha G.$$

Furthermore if we replace the operator ∂_α by $\frac{\delta}{\delta t}$ in the relation (3·16) we get the value of $\frac{\delta S_H}{\delta t}$

To obtain the expression for curvature of the magnetic line behind the shock surface we proceed as follows.

Let λ^i be the components of the unit tangent vector to the magnetic line at a point Q just behind the shock surface, then we have the relation

(3·25) $$\lambda^i = \frac{\partial x^i}{\partial s} = \frac{H^i}{H}$$

Also we have the relation

(3·26) $$\frac{\partial \lambda^i}{\partial s} = \frac{\partial^2 x^i}{\partial s^2} = K \mu^i\ ,$$

where μ^i are components of principal normal vector at Q. Form (3·26) we obtain

$$K^2 = \frac{\partial^2 x^i}{\partial s^2} \frac{\partial^2 x^i}{\partial s^2} \tag{3·27}$$

But from (3·25) and (2·26) we get

$$\frac{\partial^2 x^i}{\partial s^2} = \frac{1}{H} \frac{\partial H^i}{\partial s} - \frac{H^i H^j}{H^3} \frac{\partial H^j}{\partial s}$$

Substituting the value of $\frac{\partial^2 x^i}{\partial s^2}$ from this in (3·27) we obtain

$$K^2 = \frac{1}{H^2} \frac{\partial H^i}{\partial s} \frac{\partial H^i}{\partial s} - \frac{1}{H^4} \left(H^i \frac{\partial H^i}{\partial s} \right)^2 \tag{3·28}$$

In consequence of (3·13), the above equation gives the value of curvature k of the magnetic line just behind the shock surface.

REFERENCES

1. Sachdeva S. K. and Mishra R. S. Determination of the curvature of the Magnetic line behind a two dimensional unsteady curved shock (under publication).
2. Sachdeva, S. K. and Mishra, R. S. Determination of gradients of flow parameters, vorticity and current density behind a two-dimensional unsteady curved shock (To appear in Tensor)
3. Mishra, R. S. Determination of jump conditions across a three demensional shock in conducting gases (unpublished).

ON SPATIAL STEADY VISCOUS COMPRESSIBLE FLOWS

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ABSTRACT

In this paper, using the geometric theory of triply orthogonal spatial curves, one of them related to the stream lines, the variations of flow quantities of steady spatial viscous compressible flows along stream lines and their principal normals and binormals have been investigated. Various dynamical and kinematical relations between the flow quantities and geometrical parameters of the stream lines have derived.

1. INTRODUCTION

The intrinsic properties of fluid have been extensively studied by various authors[2-10, 13-15] in the field of fluid mechanics by the application of differential geometry, defining the three spatial orthogonal curves. Truesdell[15] has surveyed the existing literature and has outlined a simple approach to the problem of gasdynamics. Kanwal[6,7] has established the intrinsic properties of gasdynamics and magnetogasdynamics. Premkumar [9,10] has obtained intrinsic equations of relativistic gas. George[5] has correlated the differential geometry to the problems of hydrodynamics. Kapur[8] has obtained characteristic equations governing the incompressible, viscous, steady flows, in the absence of extraneous forces when isovels coincide with the streamlines. Purushotham[11] has studied the intrinsic properties of viscous, incompressible steady flows. Suryanarayan[13] has furnished by a direct method the relations existing between flow quantities when the Beltrami surfaces are a family of surfaces obtained by revolving a family of confocal hyperbolae which are orthogaonal to streamlines. Purushotham and Indrasena[12] have established the intrinsic properties of steady gas flows and have studied extensively the properties of barotropic gas flows in comparison to the incompressible steady flows. And also compatiblity conditions obtained by Barker[1] for gas flows have been transformed into intrinsic form.

In view of such an interest in the subject we have analyased the intrinsic relations of compressible viscous, steady flows when the extrinsic forces are absent for non-conducting fluids. considering the geometric properties of three spatial orthogonal curves one related to the stream lines and the other two are to their principal normals and binormals. Transforming the basic equations governing the fluids described above we observe that the variation of the pressure along the binormal to the stream line is equal to the resolved part of viscous force in that direction. The intrinsic properties of a non-viscous, steady flows considered by Truesdell[15] and Kanwal[6,7] can be deduced as special cases. Variation of stagnation enthalphy along the stream line is not uniform as in the case of non-viscous flows. The resolved parts of vorticity components along principal normal and binormal to the streamlines are obtained in terms of the flows quantities. The normals to the isobars, isopycnics, isotherms and isentropics are coplaner. General expression for the rate of circulation is obtained. Finally the variation of density along the principal normals and binormals are expressed in terms of flow quantities.

2. BASIC EQUATIONS

(*A*) The basic equations governing the steady, viscous, compressible flow in the absence of external forces, thermally non conducting with two specific heats constant are giving below in the usual notation[16].

Equation of continuity

(1) $$\operatorname{Div}(\rho \vec{q}) = 0$$

Equation of motion

(2) $$\vec{r} = \frac{d\vec{q}}{dt} = (\vec{q}\cdot\nabla)\vec{q} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\vec{q} + \frac{\mu}{3\rho}\operatorname{grad}\theta$$

$$= -\frac{1}{\rho}\nabla p + \nu\left[\nabla^2\vec{q} + \tfrac{1}{3}\operatorname{grad}\theta\right]$$

where

$$\theta = \operatorname{div}\vec{q}$$

(3) $$T\nabla S = \nabla h - \frac{1}{\rho}\nabla p$$

Equation of energy

(4) $$\vec{q}\cdot\nabla S = 0$$

Equation of state

(5) $$\rho = \rho(p, S)$$

(6) $$-\vec{q}\wedge\vec{\omega} = -\nabla h_0 + T\nabla S + \nu\left\{\nabla^2\vec{q} + \tfrac{1}{3}\operatorname{grad}\theta\right\}$$

(*B*) GEOMETRICAL RELATIONS

Following Purushotham[11] *i.e.* considering $\vec{t}$, $\vec{n}$ and $\vec{b}$ as triply orthogonal unit tangent vectors along the curves of congruences formed by streamlines, principal normals and binormals respectively and $\frac{d}{ds}, \frac{d}{dn}, \frac{d}{db}$ as directional derivatives along these vectors and selecting $\vec{r}$ as the position vector in space, we have the following geometrical relations[6,17].

(7) $$\frac{d\vec{r}}{dt} = \vec{t} = \frac{\vec{q}}{q}$$

(8) $\frac{d\vec{t}}{ds} = \vec{n}K$ (9) $\frac{d\vec{b}}{ds} = -\vec{n}\tau$ (10) $\frac{d\vec{n}}{ds} = \vec{b}\tau - \vec{t}K$

(11) $\frac{d\vec{n}}{dn} = \vec{t}K'$ (12) $\frac{d\vec{b}}{dn} = -\vec{t}\sigma'$ (13) $\frac{d\vec{t}}{dn} = \vec{b}\sigma' - \vec{n}K'$

(14) $\frac{d\vec{b}}{db} = \vec{t}K''$ (15) $\frac{d\vec{n}}{db} = -\vec{t}\sigma''$ (16) $\frac{d\vec{t}}{db} = \vec{n}\sigma'' - \vec{b}K''$

where (K, K', K'') and $(\tau, \sigma', \sigma'')$ are the curvatures and torsions of the above curves.

(C) VECTORIAL RELATIONS

(17) $\operatorname{div} \vec{t} = -(K' + K'')$ (18) $\operatorname{div} \vec{n} = -K$ (19) $\operatorname{div} \vec{b} = 0$

(20) $\operatorname{curl} \vec{t} = \vec{t}(\sigma' - \sigma'') + \vec{b} K$ (21) $\operatorname{curl} \vec{n} = -\vec{n}(\sigma'' + \tau) - \vec{b} K'$

(22) $\operatorname{curl} \vec{b} = \vec{n} K'' + \vec{b}(\sigma' - \tau)$

Using the solenoidal property of (20), (21) and (22) we obtain

$$(23) \qquad \operatorname{div} \operatorname{curl} \vec{t} = \frac{dk}{db} + \frac{d}{ds}(\sigma' - \sigma'') + (K' + K'')(\sigma'' - \sigma') = 0$$

$$(24) \qquad \operatorname{div} \operatorname{curl} \vec{n} = K(\sigma'' + \tau) - \frac{d}{dn}(\sigma'' + \tau) - \frac{dK'}{db} = 0.$$

$$(25) \qquad \operatorname{div} \operatorname{curl} \vec{b} = \frac{dK''}{dn} - KK'' + \frac{d}{db}(\sigma' - \tau) = 0$$

The vorticity vector is given by

$$\operatorname{curl} \vec{q} = \vec{\omega} = \operatorname{curl} (\vec{t} q) = q \operatorname{curl} \vec{t} + \nabla q \wedge \vec{t}$$

Using (20) this can be expressed as

$$(26) \qquad \operatorname{curl} \vec{q} = \vec{\omega} = \vec{t} q (\sigma' - \sigma'') + \vec{n} \frac{dq}{db} + \vec{b}\left(Kq - \frac{dq}{dn}\right)$$

3. TRANSFORMATION OF EQUATIONS

(A) In this section we shall transform the basic equations in intrinsic form and study some of the interesting properties.

Using (7) and (17), (1) can be expressed as

$$(27) \qquad \frac{d}{ds} \log (\rho q) = (K' + K'') = -J$$

When the surfaces are minimal *i.e.* $(K' + K'') = 0$, the flux, ρq along the stream line is constant and the density varies as the inverse of the velocity of the fluid.

Evaluating $\nabla^2 \vec{q}$, where ∇^2 is the Laplacian operator, using the vectorial identity

$$\nabla^2 \vec{q} = \operatorname{grad} \theta - \operatorname{curl} \operatorname{curl} \vec{q}$$

and equation (23) and separating the components of (2) we obtain the following intrinsic equations of momentum for steady viscous flow

(28) $\frac{1}{2}\frac{dq^2}{ds} = -\frac{1}{\rho}\frac{dp}{ds} + \nu\left\{\nabla_1^2 q - \frac{d}{dn}(Kq) - \frac{d^2q}{ds^2} - q(\sigma' - \sigma'')^2 + \frac{4}{3}\frac{d\theta}{ds}\right\}$

(29) $Kq^2 = -\frac{1}{\rho}\frac{dp}{dn} + \nu\left\{(\tau + 2\sigma'' - \sigma')\frac{dq}{db} - q\frac{d}{db}(\sigma' - \sigma'') - K''(Kq - \frac{dq}{dn}) + \frac{d}{ds}(Kq - \frac{dq}{dn}) + \frac{4}{3}\frac{d\theta}{dn}\right\}$

(30) $0 = -\frac{1}{\rho}\frac{dp}{db} + \nu\left\{K'\frac{dq}{db} + q\frac{d}{dn}(\sigma' - \sigma'') - \frac{d^2q}{ds\,db} + (Kq - \frac{dq}{dn})(\tau + \sigma'' - 2\sigma') + \frac{4}{3}\frac{d\theta}{db}\right\}$

where $\nabla_1^2 = \frac{d^2}{ds^2} + \frac{d^2}{dn^2} + \frac{d^2}{db^2}$

From the equation (30) we conclude that the variation of pressure along the binormal to the streamline varies as the resolved part of the viscous force in that direction. From equations (28) to (30) the equations derived by Kanwal[6] and Truesdell[15] can be obtained by putting $\nu = 0$. It is clear from these that the pressure is not uniform along individual binormal as in the case of non viscous flows. Also the cons ancy of velocity does not imply the constancy of pressure and curvature along individual streamlines as in the case of nonviscous flows.

Equation (6) can be expressed as

$$\text{curl}\,\vec{q} \wedge \vec{q} = -\nabla h_0 + T\nabla S + \nu\left\{\nabla^2\vec{q} + \tfrac{1}{3}\,\text{grad}\,\theta\right\}$$

Taking the Scalar product of the above equation by $\vec{t}$ and using (4) we get

(31) $$\frac{dh_0}{ds} = \nu\left\{\nabla_1^2 q - \frac{d}{dn}(Kq) - \frac{d^2q}{ds^2} - q(\sigma' - \sigma'')^2 + \frac{4}{3}\frac{d\theta}{ds}\right\}$$

which determines the variation of the stagnation enthalpy along the streamlines as the resolved part of viscous force along the streamline, it is evident that it is uniform in the case of non-viscous flows.

Similarly taking the scalar product of (6) by $\vec{n}$ we obtain

(32) $$(Kq - \frac{dq}{dn}) = \vec{w}.\vec{b} = -\frac{1}{q}\frac{dh_0}{dn} + \frac{T}{q}\frac{dS}{dn} + \frac{\nu}{q}\left\{(\tau + 2\sigma'' - \sigma')\frac{dq}{db} - q\frac{d}{db}(\sigma' - \sigma'') - K''(Kq - \frac{dq}{dn}) + \frac{d}{ds}(Kq - \frac{dq}{dn}) + \frac{4}{3}\frac{d\theta}{dn}\right\}$$

The above intrinsic equation gives the component of vorticity along the binormal to the streamline. In the case of non-viscous flows this simplifies to

(33) $$\vec{w}.\vec{b} = -\frac{1}{q}\frac{dh_0}{dn} + \frac{T}{q}\frac{dS}{dn}$$

which for homentropic flow reduces to

(34) $$\vec{w}.\vec{b} = -\frac{1}{q}\frac{dh_0}{dn}$$

It follows that in the case of non-viscous homentropic flow, the variation of stagnation enthalphy along the principal normal varies as the product of velocity and the resolved part of the vorticity along the binormal to the streamline.

Multiplying (6) scalarly by $\vec{b}$ we obtain

(35) $$\frac{dq}{db} = \vec{w}.\vec{n} = \frac{1}{q}\frac{dh_0}{db} - T\frac{dS}{db} - \frac{\nu}{q}\left\{ K'\frac{dq}{db} + q\frac{d}{dn}(\sigma' - \sigma'') - \frac{d^2q}{dsdb} + \left(Kq - \frac{dq}{dn}\right)(\tau + \sigma'' - 2\sigma') + \frac{4}{3}\frac{d\theta}{db}\right\}$$

This gives the resolved part of vorticity along the principal normal to the streamline. For non-viscous and homentropic flow (35) simplifies to

(36) $$\vec{w}.\vec{n} = \frac{1}{q}\frac{dh_0}{db}$$

from which we conclude that the variation of the stagnation enthalphy along the binormal varies as the product of velocity and the component of vorticity along the principal normal to the streamline.

(*B*) Now operating curl on (3) we obtain

(37) $$(\nabla T \wedge \nabla S) \wedge (\nabla \rho \wedge \nabla p) = 0$$

Writing (3) in the form $\rho T \nabla S = \rho \nabla h - \nabla p$ and operating curl on this and taking scalar product of the resulting equation by ∇S we get

(38) $$\nabla S \,.[\nabla \rho \wedge \nabla h] = 0$$

From (37) and (38) we have the following theorem :

Theorem : The normals to isotherms, isentropics, isopycnics isobars and equi enthalphy lines are coplanar.

(*C*) Now we shall obtain the expression for circulation in any closed circuit in the region of flow.

Let Γ be the circulation of fluid around a contour L in the region of flow, the rate of which is expressed by

$$\frac{d\Gamma}{dt} = \int_L \frac{d\vec{q}}{dt}.d\vec{r}$$

Substituting for $\frac{d\vec{q}}{dt}$ from (2) we obtain

(39) $$\frac{d\Gamma}{dt} = \int_L \left\{ T\nabla S - \nabla h + \nu(\nabla^2\vec{q} + \nabla\theta)\right\}.d\vec{r}$$

Applying Stoke's theorem, it can be written as

(40) $$\frac{d\Gamma}{dt} = \int_\sigma \int \text{curl} \left\{ T \nabla S - \nabla h + \nu (\nabla^2 \vec{q} + \nabla \theta) \right\} . \hat{m}\, d\sigma$$

Where $\hat{m}$ is the unit normal to surface.

Equation (40) gives the rate of change of circulation for non-viscous flows and this simplifies to

(41) $$\frac{d\Gamma}{dt} = \int_\sigma \int (\nabla T \wedge \nabla S) . \hat{m}\, d\sigma$$

The integrand in (41) vanishes if

(*i*) T and S are functionally related, *i e.* isotherms and isentropics are coincident.

(*ii*) The normals to surfaces T = constant, S = constant,

and $\hat{m}$ are coplanar

and (*iii*) entropy is uniform *i.e.* homentrophy.

Therefore in all the above three cases the circulation is conserved.

Operating ∇ on the equation (5) we get

(42) $$\nabla \rho = \frac{\partial \rho}{\partial p} \nabla p + \frac{\partial \rho}{\partial S} \nabla S$$

Using $\frac{\partial \rho}{\partial p} = \frac{1}{c^2}$ and $M = q/c$, (42) becomes

(43) $$\nabla \rho = \frac{M^2}{q^2} \nabla p + \frac{\partial \rho}{\partial S} \nabla S$$

where c is the velocity of sound and 'M' is the Mach number.

Taking the scalar product of (43) by $\vec{t}$ we get

(44) $$\frac{d\rho}{ds} = \frac{M^2}{q^2} \frac{dp}{ds}$$

Substituting for $\frac{d\rho}{ds}$ and $\frac{dp}{ds}$ respectively from (27) and (28) in (44) we obtain

(45) $$q^2 \left\{ (K' + K'') - \frac{d}{ds} \log (q) \right\} = M^2 \nu \left\{ \nabla_1^2 q - \frac{d}{dn} (Kq) - \frac{d^2 q}{ds^2} - q (\sigma' - \sigma'')^2 + \tfrac{4}{3} \frac{d\theta}{ds} \right\} - \frac{M^2}{2} \frac{dq^2}{ds}$$

Substituting again for $\frac{dp}{ds}$ and $\frac{dp}{ds}$ from (27) and (3) respectively in (44), and using (4) we have.

$$q^2 \{ (K' + K'') - \frac{d}{ds} \log (q) \} = M^2 \left(\frac{dh}{ds} \right) \tag{46}$$

Now taking the scalar product of (43) by $\overrightarrow{n}$ we obtain

$$\frac{d\rho}{dn} = \frac{M^2}{q^2} \frac{dp}{dn} + \frac{\partial \rho}{\partial S} \frac{dS}{dn} \tag{47}$$

Substituting for $\frac{dp}{dn}$ from (29) we get

$$\frac{d\rho}{dn} = \frac{M^2}{q^2} \rho \Bigg[\nu \Big\{ (\tau + 2\sigma'' - \sigma') \frac{dq}{db} - q \frac{d}{db} (\sigma' - \sigma'') - K'' (Kq - \frac{dq}{dn}) + \frac{d}{ds} (Kq - \frac{dq}{dn}) + \tfrac{4}{3} \frac{d\theta}{dn} \Big\} - Kq^2 \Bigg] + \frac{\partial \rho}{\partial S} \frac{dS}{dn} \tag{48}$$

In the case of homentrophic flow the above equation simplifies to

$$\frac{d\rho}{dn} = \frac{M^2}{q^2} \rho \Bigg[\nu \Big\{ (\tau + 2\sigma'' - \sigma') \frac{dq}{db} - q \frac{d}{db} (\sigma' - \sigma'') - K^r (Kq - \frac{dq}{dn}) + \frac{d}{ds} (Kq - \frac{dq}{dn}) + \tfrac{4}{3} \frac{d\theta}{dn} \Big\} - Kq^2 \Bigg] \tag{49}$$

Substituting for $\frac{dp}{dn}$ from (3) in (47) we get

$$\frac{d\rho}{dn} = \frac{M^2}{q^2} \rho \left\{ \frac{dh}{dn} - T \frac{dS}{dn} \right\} + \frac{\partial \rho}{dS} \frac{dS}{dn} \tag{50}$$

Multiplying (43) scalarly by $\overrightarrow{b}$ we obtain

$$\frac{d\rho}{db} = \frac{M^2}{q^2} \frac{dp}{db} + \frac{\partial \rho}{\partial S} \frac{dS}{db} \tag{51}$$

Substituting for $\frac{dp}{db}$ from (30) and (3) respectively we have

$$\frac{d\rho}{db} = \frac{M^2}{q^2} \rho \nu \Big\{ K' \frac{dq}{db} + q \frac{d}{dn} (\sigma' - \sigma'') - \frac{d^2 q}{ds db} + (Kq - \frac{dq}{dn}) \times (\tau + \sigma'' - 2\sigma') + \tfrac{4}{3} \frac{d\theta}{db} \Big\} + \frac{\partial \rho}{\partial s} \frac{dS}{db} \tag{52}$$

$$\frac{d\rho}{db} = \frac{M^2}{q^2} \rho \left\{ \frac{dh}{db} - T \frac{dS}{db} \right\} + \frac{\partial \rho}{\partial S} \frac{dS}{db} \tag{53}$$

From (52) it follows that for non viscous and homentropic flow the variation of density along the binormal to the streamline is uniform.

REFERENCES

1. Berker, R. Comptes, Rendues des, Academie des Sciences de Paris, 1342, (1956).
2. Byushgens, S. S. *Acad. Nauk. Maths. Series*, 12 : 481, (1948).
3. Coburn N. *Michigan Maths. Jr*, 1 : 113, (1952).
4. Cristea, Ion. J. *C. R. Acad. Sci. Paris*, 253 : 2843, (1961).
5. George, Vg. *Bulg. Akad. Nauk Izv. Mat. Inst.*, 6 : 101, (1962)
6. Kanwal, R. P. *Jr. of Maths. and Mech* 6 : 621, (1957).
7. Kanwal, R. P. *ZAMM. B.* 41 : 462, (1961).
8. Kapur, J. N. *Bull. Cal. Maths. Soc* 53 : 95, (1961).
9. Premkumar, *Tensor (N. S.)* 12 : 239, (1962).
10. Premkumar, *Tensor (N S)* 12 : 65, (1962).
11. Purushotham, G. *App. Sci. Res Sec* A 15 : 23, (1965).
12. Purushotham, G and Indrasena, A. *App Sci Res. Sec A*, (*to appear in Vol.* 16)
13. Suryanarayan E. R. *Jr. Maths. and Mech.* 13 : 164, (1964).
14. Thomas, T. Y. *Comm. Pure and App. Maths.* 3 : 103, (1950).
15. Truesdell, C. *ZAMM. B.* 40 : 9, (1960).
16. Vazsonyi, A. *Qly. of App. Maths.* 3 (1), 29, (1945).
17. Weatherburn, C. E. *Diff. Geo. of three dimensions*, p. 15 : *Camb. Univ. Press*, (1955

ON KINEMATICAL ASPECTS OF MAGNETOGASDYNAMICAL FLOWS

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ABSTRACT

Considering the geometry of the spatial curves related to the streamlines and assigning unit constant vector in the direction of magnetic field the kinematical and dynamical properties of the hydromagnetic flows arrived at are :

(*i*) Decomposing the momentum equations into intrinsic form it is observed that the magnetic pressure is uniform along the binormal to the streamlines.

(*ii*) The compatibility conditions to be satisfied by the velocity vector and the magnetic fields are expressed in terms of the curvatures and torsions related to the streamlines.

(*iii*) (*a*) Defining Bernoulli Surfaces it is proved that these contain both streamlines and vortex lines as in the case of non-hydromagnetic flows ; also the necessary and sufficient condition, for these surfaces to be parallel, is established.

(*b*) The existence of Bernoulli Surfaces in the case of an incompressible flow is discussed and they constitute a family of parallel surfaces defined by $P + E =$ Const, where P and E are the hydromagnetic pressure and the sum of the magnetic and kinetic energies.

(*iv*) Uniformity of either magnetic pressure or the velocity along the streamlines implies the magnetic field to decrease and the hydrodynamic pressure to increase along the individual streamlines

(*v*) For a Beltrami flow, it is proved that the streamlines are either circular helices or circles or parallel lines, also the magnitude of the velocity, curvature, torsion of the streamlines and the magnetic pressure are uniform in the rectifying plane The kinematical conditions to be satisfied by these flows are deduced.

1. INTRODUCTION

In this paper, we shall be concerned with an electrically conducting, ideal, compressible fluid, devoid of viscosity and thermal conductivity. We shall assume further that the (mean) electric charge is zero (so that the medium is essentially neutral), that the displacement currents may be neglected and that the electrical conductivity of the fluid is infinite. Under these assumptions, commonly made in hydromagnetics, the basic equations governing the fluid flow can be written as[1,2]

$$\text{div}\,(\rho\,\vec{q}) = 0 \tag{1}$$

$$\rho\,(\vec{q}\,.\,\nabla)\,\vec{q} + \nabla\,p - \eta\,(\vec{H}\,.\,\nabla)\,\vec{H} = 0 \tag{2}$$

$$(\vec{q}\cdot\nabla)\,\vec{H} - (\vec{H}\,.\,\nabla)\,\vec{q} + \vec{H}\,\text{div}\,\vec{q} = 0 \tag{3}$$

$$\text{div}\,\vec{H} = 0 \tag{4}$$

$$\vec{q}\,.\,\nabla\,S = 0 \tag{5}$$

$$p = \rho^{\gamma}\,e^{S/Jc_v} \tag{6}$$

Where 't' denotes the time, p the fluid pressure, ρ the density, $\vec{q}$ the velocity vector, $\vec{H}$ the magetic field vector, $P = p + \eta \frac{H^2}{2}$ the magnetic pressure, S the entropy, γ the adiabatic exponent, J_{cv} the Joule constant and μ_e the magnetic permeability.

Equation (1) is the continuity equation of the fluid-dynamics, relations (2) are the equations of motion, the last two terms on the left hand side of (2) together represents the Lorentz force per unit mass. The two Maxwell equations which do not involve current and charge have been written as (3) and (4). In writing (3) in this form we have made use of (4). Equations (5) and (6) express the energy and the state relations to be satisfied by a flow.

§ 2. GEOMETRICAL AND VECIORIAL RELATIONS

Considering $\vec{t}$, $\vec{n}$ and $\vec{b}$ as triply unit tangent orthogonal vectors along the curves of congruences formed by streamlines, principal normals and binormals respectively and denoting $\frac{d}{ds}$, $\frac{d}{dn}$, $\frac{d}{db}$ as directional derivatives along these vectors, also selecting $\vec{r}$ as the position vector in space, the following geometrical and vectorial relations are obtained by Purushotham[4]

$$\frac{dr}{ds} = \vec{t} = \frac{\vec{q}}{q} \tag{7}$$

(8) $\frac{d\vec{t}}{ds} = \vec{n} K$ (9) $\frac{d\vec{b}}{ds} = -\vec{n}\tau$ (10) $\frac{d\vec{n}}{ds} = \vec{b}\tau - \vec{t} K$

(11) $\operatorname{div} \vec{t} = -(K' + K'') = J_1$ (12) $\operatorname{div} \vec{n} = -K$ (13) $\operatorname{div} \vec{b} = 0$

(14) $\operatorname{curl} \vec{t} = \vec{t}(\sigma' - \sigma'') + \vec{b} K$ (15) $\operatorname{curl} \vec{n} = -\vec{n}(\tau + \sigma'') - \vec{b} K$

$$\operatorname{curl} \vec{b} = \vec{n} K'' + \vec{b}(\sigma' - \tau) \tag{16}$$

Using (7) and (14), we can express the vorticity vector as

$$\operatorname{curl} \vec{q} = \vec{\zeta} = \vec{t} q(\sigma' - \sigma'') + \vec{n}\frac{dq}{db} + \vec{b}\left(K q - \frac{dq}{dn}\right) \tag{17}$$

where (K, K', K'') and $(\tau, \sigma', \sigma'')$ are the curvatures and torsions of the above defined curves.

We shall consider the magnetic field to be unidirectional, the unit vector $\vec{h}$ in this direction to be constant *i.e.*

$$\vec{H} = \vec{h} H \tag{18}$$

§ 3. DECOMPOSITION INTO INTRINSIC FORM

Let us define the operator gradient (∇) as

$$\nabla = \vec{t}\,\frac{d}{ds} + \vec{n}\,\frac{d}{dn} + \vec{b}\,\frac{d}{db} \tag{19}$$

Now using (7) and (11), we can transform (1) into intrinsic form as

$$\frac{d}{ds}\log(\rho q) = (K' + K'') = -J_1 \tag{20}$$

where J_1 is the mean curvature of the normal surfaces of the streamlines. This has been obtained by Suryanarayan[5] and Purushotham[4].

Using (18) in (4) we obtain

$$\frac{dH}{dh} = 0 \tag{21}$$

where $\frac{d}{dh}$ is the directional derivative along the unit constant vector $\vec{h}$. The result (21) has been observed by Suryanarayan.[5]

Using (18), (19), (21) and the above geometrical and vectorial relations in equation (2) and taking the scalar product of the resultant equation by $\vec{t}$, $\vec{n}$ and $\vec{b}$ we have

$$\rho q\,\frac{dq}{ds} + \frac{dP}{ds} = 0 \quad (22) \qquad K\rho\, q^2 + \frac{dP}{dn} = 0 \quad (23) \qquad \frac{dP}{db} = 0 \quad (24)$$

This approach is more elegant than the one considered by Suryanarayan[5]. From (22) we observe that the uniformity of the magnetic pressure along the stramline implies the uniformity of the velocity along the individual streamline and the converse. If the magnetic pressure is uniform in the principal plane, then the streamlines are straight.

Multiplying (3) scalarly by $\vec{t}$, $\vec{n}$ and $\vec{b}$ and after a little simplification, we obtain the following relations.

$$q\,\frac{dH_t}{ds} = H_n\left(Kq + \frac{dq}{dn}\right) + H_b\,\frac{dq}{db} + H_t\,(K' + K'')\,q \tag{25}$$

$$\frac{d}{ds}(q\,H_n) = H_b\,(\tau + \sigma'')\,q + q.K''\,H_n \tag{26}$$

$$\frac{d}{ds}(q\,H_b) = H_n\,(\sigma' - \tau)\,q + q\,K'\,H_b \tag{27}$$

Where H_t, H_n, H_b are the resolved parts of the magnetic field along the streamlines and their principal normals and binormals respectively.

Eliminating 'q' from (26) and (27) we obtain

$$\frac{d}{ds}\left(\frac{H_n}{Hb}\right) = \left(1 + \frac{Hn^2}{H_b^2}\right)\tau + \left(\sigma'' - \sigma'\,\frac{H_n^2}{H_b^2}\right) + \frac{H_n}{H_b}\,(K'' - K') \tag{28}$$

The equations (25) and (28) constitute the compatib lity conditions to be satisfied by a magnetic field and velocity of the fluid flow.

§ 4. GEOMETRY OF THE BERNAULLI SURFACE

The equation (2) on using the vectorial identities can be expressed as

$$-\rho \operatorname{curl} \vec{q} \wedge \vec{q} + \vec{\eta} \operatorname{curl} \vec{H} \wedge \vec{H} = \nabla P + \frac{\rho}{2} \nabla q^2 - \frac{\eta}{2} \nabla H^2 \tag{29}$$

which on using (7) and (18) together with geometrical and vectorial relation can be written as

$$-\rho q \vec{b} \frac{dq}{db} + \vec{n} \rho q \left(K q - \frac{dq}{dn} \right) = - \nabla P - \frac{\rho}{2} \nabla q^2 \tag{30}$$

Let us define the Bernoulli Surface as

$$\nabla B = - \frac{1}{\rho} \{ \nabla P + \tfrac{1}{2} \nabla (\rho q^2) \} + \frac{q^2}{2\rho} \nabla \rho \tag{31}$$

Using (30) and (31) we obtain

$$\nabla B = - \vec{b} q \frac{dq}{db} + \vec{n} q \left(K q - \frac{dq}{dn} \right) \tag{32}$$

from which we conclude that the normals to the Bernoulli surfaces lie in the normal plane of the streamlines, *i. e.* the Bernoulli surfaces contain the streamlines.

Using (17) and (32) we have

$$\nabla B . \operatorname{curl} \vec{q} = 0 \tag{33}$$

which shows that the vortex lines lie on the Bernoulli surface. This property is independent of magnctie field.

Also considering $\frac{d}{dN}$ as the directional derivative along the normal to the Bernoulli surface, which lies in the normal plane, we have

$$\frac{dB}{dN} \cos \alpha = \frac{dB}{dn} = q \left(K q - \frac{dq}{dn} \right) \tag{34}$$

$$\frac{dB}{dN} \sin \alpha = \frac{dB}{db} = - q \frac{dq}{db} \tag{35}$$

where 'α' is the angle between the normal to the Bernoulli surface and the principal normal to the streamline

From (34) and (35) we obtain

$$\frac{dB}{dN} = q \left\{ \left(\frac{dq}{db} \right)^2 + \left(K q - \frac{dq}{dn} \right)^2 \right\}^{\frac{1}{2}} = q \left(\zeta^2_n + \zeta^2_b \right)^{\frac{1}{2}} \tag{36}$$

where ζ_n and ζ_b are the resolved parts of the vorticity $\vec{\zeta}$, along the principal normal and binormal to the streamline respectively.

From (36) we infer that the necessary and sufficient condition for a system of Bernoulli surface to be parallel is

$$q^2 \left\{ \left(\frac{dq}{db} \right)^2 + \left(Kq - \frac{dq}{dn} \right)^2 \right\} = \text{constant} \tag{37}$$

along each surface of the family. This has been proved by Suryanarayan[5] by a circuitous method.

When we consider incompressible fluid flows, (31) simplifies to

$$\nabla B = -\frac{1}{\rho}\nabla\left(p + \frac{q^2}{2} + \eta\frac{H^2}{2}\right) = -\frac{1}{\rho}\nabla(p+E) \tag{38}$$

where E is the sum of the magnetic and kinetic energies. Equation (38) is exactly integrable, consequently the Bernoulli surfaces in incompressible flow exist, and they constitute a family of surfaces $p + E =$ Const.

Using (6) in (38) we get

$$\nabla B = \frac{1}{\rho}\left\{\rho^{\gamma}\ \nabla e^{S/J c v} + e^{S/J c v}\nabla\rho^{\gamma} + \nabla E\right\} \tag{39}$$

which on scalar multiplication by $\vec{n}$ and $\vec{b}$ and using (32) we have

$$\zeta_n = \frac{1}{q}\left\{\frac{c^2}{\gamma J c v}\frac{dS}{db} + \frac{c^2}{\rho}\frac{d\rho}{db} + \frac{1}{\rho}\frac{dE}{db}\right\} \tag{40}$$

$$\zeta_b = -\frac{1}{q}\left\{\frac{c^2}{\gamma J c v}\frac{dS}{dn} + \frac{c^2}{\rho}\frac{d\rho}{dn} + \frac{1}{\rho}\frac{dE}{dn}\right\} \tag{41}$$

These express the vorticity components along the principal normal and binormal to the streamlines in terms of entropy, density and kinetic energy.

Equating $\frac{d}{ds}$ log ρ from (1) and (2) we obtain

$$-\frac{d}{ds}\log\rho = \frac{1}{q}\operatorname{div}\vec{q} = \frac{1}{c^2}\left\{\vec{t}\cdot(\vec{q}\cdot\nabla)\vec{q} + \frac{\eta H}{\rho}\vec{t}\cdot\nabla H\right\} \tag{42}$$

Now using (7) and (11) in (42) we get

$$q\frac{dq}{ds} = \frac{1}{(1-M^2)}\left\{q^2(K' + K'') + \eta\frac{H}{\rho}M^2\frac{dH}{ds}\right\} = -\frac{1}{\rho}\frac{dP}{ds} \tag{43}$$

$$= -\frac{1}{\rho}\left(\frac{dp}{dS} + \eta H\frac{dH}{ds}\right)$$

From this it follows that the uniformity of the magnetic pressure along individual streamline implies the constancy of velocity and the converse, in which case the magnetic field decreases and the hydrodynamic pressure increases along the streamline.

§ 5. BELTRAMI FLOW

The Beltrami flow is defined by[3] as

$$\operatorname{curl}\vec{q} = \lambda\vec{q} \tag{44}$$

which, on using (17) yields the following relations

(45) $\lambda = \sigma' - \sigma''$ (46) $\frac{dq}{db} = 0$ (47) $Kq - \frac{dq}{dn} = 0$

Using (46) in (47) we obtain

$$\frac{dK}{db} = 0 \tag{48}$$

Following Weatherburn[6] we have

$$a\,(K^2 + \tau^2) = K \tag{49}$$

which on using (48) yields

$$\frac{d\tau}{db} = 0 \tag{50}$$

Operating divergence on (44) and using the solenoidal properties of vorticity vector and (14) we obtain the following relations :

$$\frac{dq}{dS} = 0 \tag{51}$$

$$\frac{d}{ds} \log (\sigma' - \sigma'') = K' + K'' \tag{52}$$

Using (51) in (47) and (49) we have

$$\frac{dK}{ds} = 0 \tag{53}$$

$$\frac{d\tau}{ds} = 0 \tag{54}$$

From (46), (48), (50), (51), (53) and (54) it follows that the magnitude of the velocity, curvature and the torsion of the streamline are constant in the rectifying plane.

From (53) and (54) we infer that the streamlines are either right circular helices or circles or parallel lines[4]

Relation (20) on using (51) and (52) gives

$$K' + K'' = \frac{d}{ds} \log \rho = \frac{d}{ds} \log (\sigma' - \sigma'') \tag{55}$$

which on integration along individual steamline simplifies to

$$\rho = c\,(\sigma' - \sigma'') \tag{56}$$

where 'C' is constant along individual streamline. Using (51) in (22) we obtain

$$\frac{dP}{ds} = 0 \tag{57}$$

This together with (24) shows that the magnetic pressure is uniform in the rectifying plane of the streamlines.

Relations (25), (27), on using (46) and (51) simplify to

$$\frac{dH_t}{ds} = 2\,KH_n + H_t\,(K' + K'') \tag{58}$$

$$\frac{dH_n}{ds} = H_b\,(\tau + \sigma'') + H_n\,K'' \tag{59}$$

$$\frac{dH_b}{ds} = H_n\,(\sigma' - \tau) + K'\,H_b \tag{60}$$

These express the variation of the resolved parts of the magnetic field along the streamlines and their principal normals and binormals respectively, along individual streamline for a Beltrami flow.

REFERENCES

1. Cowling, T. G. Magneto hydrodynamics, *Interscience Publishers, N. Y.* (1957).
2. Kanwal, R. P. *Arch. Rat. Mech. Anal.* 4 : 335, (1960).
3. Milne-Thomson. *Theoretical Hydrodynamics*, 76, (1955).
4. Purushotham, G. *Appl. Sci. Res. A.* 15 : 23, (1965).
5. Suryanarayan, E. R. *Proc. Amer. Maths, Soc.* 16 : 90, (1965).
6. Weatherburn, C. E. Diff. Geo. Three Dimens. Camb. *Univ. Press*, (1955).

EFFECT OF SODIUM HUMATE ON THE YIELD AND NITROGEN CONTENT OF RYE GRASS (*LOLIUM PERENNE*)

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ABSTRACT

The influence of sodium humate prepared from farmyard manure was investigated on the yield and nitrogen content of Rye grass (*Lolium perenne*) under greenhouse conditions. The yield, nitrogen content and uptake of nitrogen by the crop was significantly increased in the presence of 0 05 and 0·25 per cent sodium humate. The number of total bacteria was almost doubled over control due to the addition of 0·05 per cent sodium humate to the soil.

Soil organic matter plays a direct role in soil fertility as a source of mineral nutrients for plants, liberated in available form in the course of mineralization. Indirectly it influences soil structure, facilitates drainage and aeration, increases water holding capacity of soils and modifies largely the physico-chemical properties, buffer capacity and cation exchange capacity.

The direct effect of humic acid on plants has been studied by Gaur[1] Christeva[2] and Chaminade[3] During the previous investigations in this laboratory an increase of 40 per cent over control in the root length of cress seeds (*Lepidium sativum*) with sodium humate having a concentration of 10^{-1}g per litre was observed. Gaur and Mathur[4] showed that addition of very small amounts of humic substances to Jensen's liquid medium caused significant stimulating effect on the nitrogen fixation by *Azotobacter chroococcum*. This specific effect of humic substances has been little studied although it appears to be important in physiological and biochemical processes of plants and soil microorganisms. The present investigation deals with the effect of sodium humate on the yield and nitrogen content of Rye grass.

MATERIALS AND METHODS

Soil was collected from I. A. R. I. Farm, New Delhi, for the experiments. Analysis of the soil is presented in table 1.

TABLE 1

Carbon	0·311%
Nitrogen	0·044%
C/N ratio	7·5 : 1
Humic acid	0·044%
Total P_2O_5	0·063%
Available P_2O_5	0·017%
Total K_2O	0·631%
Available K_2O	0·0052%
Water holding capacity	37·5%
pH	7·5

Humic acid was extracted from farmyard manure by 3 per cent ammonium oxalate Chaminade[5]. This was washed with distilled water to remove oxalate and chloride ions. To prepare sodium humate N/10 sodium hydroxide was added slowly until pH 7 was attained Excess of alkali and other minerals were dialysed against distilled water. Sodium humate was mixed with the soil so as to make 0·05% and 0·25% of it in treated soils.

One Kg. soil with or without sodium humate was transferred to glazed pots. The following nutrients were mixed with the soils in all the pots.

KH_2PO_4	0·383 gm./pot
$KHCO_3$	1·383 gms /pot
$MgSO_4.7H_2O$	0 500 gm./pot

Micro-nutrients

Solution A

$MnSO_4$	0·750 gm./litre
$CuSO_4 . 5H_2O$	0·075 gm./litre
$ZnSO_4$	0·050 gm./litre

Solution B

H_3BO_3	0·150 gm./litre
$(NH_4)_2MoO_4$	0·050 gm./litre

2 ml. each of *A* and *B* solutions of micronutrients were mixed in each pot. The chemicals used in the experiment and for preparation of humic acid and sodium humate were of A.R. quality. The humidity in the pots was maintained at half of the water holding capacity of the soil. Each treatment was quadreplicated and the results are the averages of the four replications.

1000 grains of rye grass were sown in each pot. After one week of growth, 100 mg. of nitrogen as calcium nitrate solution was added to each pot every week before the addition of water. Two cuttings were taken and dry weight of each cutting was recorded. The plant samples were analysed for total organic nitrogen by micro-Kjeldahl method. The soil samples were taken from each pot for total bacterial count. It was analysed by standard plate count method using Thornton's[6] mannitol-salt method. The composition of the medium is as follows:

K_2HPO_4	1·0 gm.	KNO_3	0 5 gm.
$MgSO_4.7H_2O$	0·2 gm.	Asparagine	0·5 gm.
$CaCl_2$	0·1 gm.	Mannitol	1 0 gm.
NaCl	0·1 gm.	Agar	14·0 gms.
$FeCl_3$	0·002 gm.	Water	1/litre

Analysis of soil was done by the methods given by Piper[7] and Jackson[8]. Humic acid of the soil was determined by Chaminade's method (*loc. cit.*).

RESULTS

TABLE 2

Dry Weights of Cuttings of Rye grass (gms.)

Treatments	1st cutting	2nd cutting	Total Yield	% increase over control
Control	1·17	1·25	2·42	–
0·05% Na-humate	1·62	1·40	3·02	24·0
0·25% Na-humate	1·40	1·37	2·77	14·4
Significant at 1% level			C.D. at 5% = 0 303	

TABLE 3

Organic nitrogen % in cuttings

Treatments	1st cutting	2nd cutting	Average of cuttings	% increase over control
Control	5·58	5·57	5·575	–
0·05% Na-humate	5·75	5·70	5·725	2·6
0·25% Na-humate	6·52	5·66	6·090	9·0

TABLE 4

Uptake of nitrogen (mgs.) by plant

Treatments	1st cutting	2nd cutting	Total of cuttings	% increase over control
Control	65·280	69·625	134·905	–
0·05% Na-humate	93·150	79·800	172·950	28·2
0·25% Na-humate	91·000	77·542	168·542	24 9
Significant at 5% level			C.D. at 5% = 37·77	

TABLE 5

Total bacterial count of soils after 2nd cutting

Treatments	Total bacterial numbers/gm. soil (Average of 4 replications)
Control	$11·0 \times 10^5$
0·05% Na humate	$21 9 \times 10^5$
0·25% Na-humate	$6·8 \times 10^5$

DISCUSSION

The foregoing results show that with the addition of sodium humate in the soil the yield of Rye grass is increased significantly over control. An increase in yield of 24·7 and 14·4 per cents over control was registered with the treatments of 0·05 and 0·25% sodium humate respectively.

Analysis of two cuttings shows that the percentage of organic nitrogen increases with an increase of the amount of sodium humate in the soil. Nitrogen uptake by the plant also increased in both the cuttings. There were significant increases of 28·2 and 24·9 per cents over control with 0·05% and 0·25% sodium humate respectively. Gaur (*loc. cit.*) has observed that in the presence of humic acid, the yield of rye grass increased significantly as well as the absorption of nitrogen, phosphorus, potassium and magnesium.

Christeva (*loc. cit.*) believes that humic acid entering the plant at the early stages of development is supplementary source of polyphenols. Humic acids, because they contain quinone groups, act as hydrogen acceptors and activators of oxygen. Guminski[9] observed that humic acid regulates the oxidation-reduction condition of the medium in which the plant is growing. He found that during oxygen deficiency humates facilitate plant respiration. This stimulating effect of humic substances may be attributed to their participation in oxidation-reduction processes of plants. This probably increases the living activity of the plant resulting in higher nitrogen uptake and yield.

The number of total bacteria was found to be higher in soils treated with 0·05% sodium humate than control. However, the numbers decreased with 0·25% humate. It seems that a higher level of sodium humate has a depressing effect on the growth of bacteria due to deterioration of physical properties of soil particularly, aeration.

SUMMARY

Due to application of 0·05% and 0·25% sodium humate to the soil, the yield of rye grass (*Lolium perenne*) registered significant increases of 24·7 and 14·4% over control respectively. However no significant difference in yield between the two levels of sodium humate treatment was recorded. The addition of sodium humate to soil caused an increase in the organic nitrogen percentage of both the cuttings. The total uptake of nitrogen by the plants was significantly increased over control in the presence of sodium humate in the soil. The total bacterial count of the soil treated with 0·05% sodium humate was higher than control soil.

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REFERENCES

1. Gaur, A. C. *Bull. Assoc. Francaise l'etude sol.*. 5 : 207–219, (1964).
2. Christeva, A. *Trans. Int. Congr. Soil. sci.*, 2 : 46–50, (1958).
3. Chaminade, R. *Trans. Inst. Congr. Soil sci.* 4 : 443–448, (1956).
4. Gaur, A. C. and Mathur, R. S. *Science and Culture*, 23 (6) : 319, (1966).
5. Chaminade, R. *Ann. Agron.*, 16 : 117–152, (1946).
6. Thornton, H. G. *Ann. Appl. Biol.*, 9 : 241–274, (1922).
7. Piper, C. S. Soil and plant analysis. *Interscience Publishers, Inc. N. Y.* (1950).
8. Jackson, M. L. Soil chemical analysis. *Asia Publishing House, Bombay*, (1962).
9. Guminski, S. *Pochvovedenia*, 12 : (1957).

ON POLYTROPIC MODELS OF UNIFORM DENSITY

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ABSTRACT

Solutions of the Lane-Emden equation of index zero has generally been regarded as a sphere of uniform density (Eddington A. S., 1926) In this paper, it has been shown that both $n=0$ and $n=-1$ correspond to spheres of uniform density. It has further been shown that the known solution for $n=$' governs a polytrope in which no pressure-density relation subsists. Equations governing polytropic models of uniform density in which pressure and density are related, have been pointed out.

INTRODUCTION

Equations governing the equilibrium are (Chandrashekhar S., 1939)

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho \quad ; \quad \frac{dM(r)}{dr} = 4\pi r^2 \rho, \tag{1}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho. \tag{2}$$

Substitutions

$$\rho = \lambda\theta^n \quad ; \quad P = K\rho^{1+1/n} = K\lambda^{1+1/n}\theta^{n+1}, \tag{3}$$

transform (2) into

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \tag{4}$$

where

$$r = \alpha\xi \quad ; \quad \alpha = \left[\frac{(n+1)K\lambda^{1/n-1}}{4\pi G}\right]^{\frac{1}{2}} \tag{5}$$

Solution of (4) for $n=0$ is

$$\theta = D + \frac{C}{\xi} - \frac{\xi^2}{6}, \tag{6}$$

where G and D are constants of integrations. The solution with the central θ unity is given by

$$\theta = 1 - \frac{\xi^2}{6}. \tag{7}$$

POLYTROPES OF UNIFORM DENSITY

Clearly, for the study of the configurations, governed by equations in (1), it is necessary to assume a relation between P and ρ; but we further see that if we put ρ=constant$=\lambda$, say in equations in (1), then no inconsistency with equations in (1) arises. Equation (2) will then be the same as equation (4) for $n=0$, with $\theta=P$ and $\alpha=(4\pi G\lambda^2)^{-1/2}$. Thus we see that if we put $\rho=\lambda$ directly in equation (2) then we get a differential equation which gives P as a function of r in a region where $\rho \neq f(r)$. This shows that we get a solution which governs a polytrope in which $P \neq f(\rho)$. Indeed, no meaning can be attached to a polytrope in which $P \neq f(\rho)$. We now therefore review such models as follows :

Let us put the pressure-density relation in the form :

$$\rho = \left[\frac{P}{K} \right]^{n/(n+1)} \tag{8}$$

The above relation is meaningless for $n = 0$ and $n = -1$, but remains meaningful for n tending to zero and n tending to minus one. For n tending to minus one, ρ tends to infinite for $P/K > 1$ and tends to zero for $P/K < 1$. Such spheres cannot be of much physical significance ; but for P/K tending to unity, the right-hand side of (8) remains indeterminate for $n \rightarrow -1$. and relation (8) can be taken relevant. For n tending to zero, ρ tends to unity for all finite values of P/K. Thus we see that both n tending to zero and to minus one correspond to spheres of uniform density.

We however have some difficulty in getting mathematically relevant solutions for n tending to 0 and -1. Author has recently discussed these solutions in detail and has shown (1966) that for n tending to 0 and -1, P and ρ remain functions of r but the exact nature of these functions remain indeterminate. He has further shown that mathematically relevant solutions for n tending to zero and to minus one are

$$v = \frac{2}{(2-u) + 2c(u-3)^{2/3} u^{1/3}}, \tag{9}$$

and

$$v = c'u \tag{10}$$

respectively where c and c' are constants of integrations and u and v are defined as

$$u = -\frac{4\pi G}{K^{2n/(n+1)}} \frac{rP^{2n/(n+1)}}{P'} ; \quad v = -\frac{rP'}{P} \tag{11}$$

Equations (9) and (10) govern spheres of uniform density. In the sphere governed by (9), relation (8) suggests that ρ tends to unity, and pressure density relation in (3) suggests that pressure, when ρ is unity, remains indeterminate. In the sphere governed by (10), P tends to K at every point and the value of density remains indeterminate. In these spheres, dP/dr and $d\rho/dr$ are infinitsimally small at the centre, and gradually tend to zero, and vanish at the boundary. That is at every point dP/dr and $d\rho/dr$ tend to vanish. The positive quadrant of $(u ; v)$ plane contains only parts of the solutions which correspond to $P > 0$ and $P' < 0$.

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REFERENCES

1. Eddington, A. S. Internal Constitution of Stars, p. 87 (1929).
2. Chandrashekhar, S. Stellar Structure, p. 87—92, (1939).
3. Srivastava, Shambhunath. A New Concept of Structure of Polytropic Configurations. *Proc. Nat. Acad. of Sci. India*, **36A** (4) (1966).

A THEOREM ON WEYL (FRACTIONAL) INTEGRAL

By

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Varma[4] gave a generalisation of the Laplace Transform

$$f(p) = \int_0^\infty e^{-px}\, h(x)\, dx \tag{1}$$

in the form

$$f(p) = \int_0^\infty e^{-\frac{1}{2}px}\, (px)^{m-\frac{1}{2}}\, W_{k,m}(px)\, h(x)\, dx \tag{2}$$

since $\qquad W_{\frac{1}{2}-m,m}(x) \equiv x^{m+\frac{1}{2}}\, e^{-\frac{1}{2}x}$

(2) reduces to (1) when $k = \frac{1}{2} - m$

In this paper we shall establish a theorem involving relation between Varma, Laplace and Weyl (Fractional) Integral defined as

$$f(p) = \frac{1}{\sqrt{\mu}} \int_p^\infty (x-p)^{\mu-1}\, h(x)\, dx \tag{3}$$

Few Corollaries of the theorem are given and further this theorem has been illustrated with suitable examples.

Throughout this paper we shall use symbollic notations

$$f(p) \doteqdot h(x)$$

$$f(p) \, \frac{V}{\overline{\overline{k,\, m}}} \, h(x)$$

and $\qquad f(p) \overset{\omega}{\underset{\mu}{=}} h(x)$

to denote (1) (2) and (3).

Theorem 1.

If $$g(p) \doteqdot h(t) \tag{2·1}$$

and $$\phi(p) \overset{\omega}{\underset{\mu}{=}} t^{n-\lambda}\, g(t) \tag{2·2}$$

then $$\phi(p) \, \frac{V}{\overline{\overline{\frac{1}{2}(1-\mu-\lambda),\, \frac{1}{2}(\mu-\lambda)}}} \, t^{\lambda-\mu}\, h^n(t) \tag{2·3}$$

provided that Laplace Transform of $|\, h(t)\,|$, Weyl (Fractional) Integral of $t^{n-\lambda}\, g(t)\,|$ and Varma Transform of $|\, t^{\lambda-\mu}\, h^n(t)\,|$ exist, $Re\ \mu > 0$, $Re\ p > 0$ and $h'(0) = h''(0) = \ldots\ldots = h^{n-1}(0) = 0$ where $h^n(t)$ denotes the n^{th} differential co-efficient of $h(t)$.

Proof: Applying well known property of Laplace Transform (1, p. 129) that if $g(p) \doteq h(t)$

then $p^n\, g(p) \doteq h^n\,(t)$ (2·4)

provided $h'(0) = h''(0) = \ldots = h^{n-1}(0) = 0$

and if (1, p. 294)

$$t^{\mu-1}\,(t+a)^{-\lambda} \doteq \frac{p^{-\mu}\, a^{-\lambda}}{\sqrt{\lambda}}\, E\,(\mu, \lambda : : ap) \tag{2·5}$$

$$|\arg a| < \pi,\ Re\,\mu > 0,\ Re\,p > 0$$

then

$$(t-a)^{\mu-1}\, t^{-\lambda}\, H(t-a) \doteq e^{-ap}\, \frac{p^{-\mu}\, a^{-\lambda}}{\sqrt{\lambda}}\, E(\mu, \lambda : : ap) \tag{2·6}$$

where $H(t)$ is Heaviside's Function.

Using (2·4) and (2·6) in Parsvel Goldstein Theorem, we get

$$\int_0^\infty t^{n-\lambda}\,(t-a)^{\mu-1}\, g(t)\, H(t-a)\, dt = \frac{a^{-\lambda}}{\sqrt{\lambda}} \int_0^\infty e^{-at}\, t^{-\mu}\, E(\mu, \lambda : : at)\, h^n(t)\, dt$$

$$\int_a^\infty t^{n-\lambda}\,(t-a)^{\mu-1}\, g(t)\, dt = \frac{a^{-\lambda}}{\sqrt{\lambda}} \int_0^\infty e^{-at}\, t^{-\mu}\, E(\mu, \lambda : : at)\, h^n(t)\, dt$$

Now with the help of the result

$$E(\mu, \lambda : : x) = \sqrt{\mu}\, \sqrt{\lambda}\, e^{\frac{1}{2}x}\, x^{-\frac{1}{2}(1-\mu-\lambda)}\, W_{\frac{1}{2}(1-\mu-\lambda),\, \frac{1}{2}(\mu-\lambda)}(x)$$

and replacing a by p we get the required result.

Corollary I. On taking $\lambda = 0$ in the theorem, it can be stated as

If $g(p) \doteq h(t)$

and $\phi(p) \overset{\omega}{\underset{\mu}{=}} t^n\, g(t)$

then $\phi(p) \doteq t^{-\mu}\, h^n\,(t)$ (2·7)

provided that Laplace Transform of $|h(t)|$, $|t^{-\mu}\, h^n\,(t)|$ and Weyl (Fractional) Integral of $|t^n\, g(t)|$ exist, $Re\,\mu > 0$, $Re\,p > 0$ and $h'(0) = h''(0) = \ldots = h^{n-1}(0) = 0$ where $h^n(t)$ denotes n^{th} differential coefficient of $h(t)$

Corollary II. On taking $n = 0$, theorem takes the following form

If $g(p) \doteq h(t)$

and $\phi(p) \overset{\omega}{\underset{\mu}{=}} t^{-\lambda}\, g(t)$

then $\phi(p) \overset{V}{\underset{\frac{1}{2}(1-\mu-\lambda),\, \frac{1}{2}(\mu-\lambda)}{=\!=\!=}} t^{\lambda-\mu}\, h(t)$ (2·8)

provided that Laplace Transform of $|h(t)|$, Weyl (Fractional) Integral of $|t^{-\lambda} g(t)|$ and Varma Transform of $|t^{\lambda-\mu}\, h(t)|$ exist, and $Re\,\mu > 0$, $Re\,p > 0$.

Example 1.

If we take [1, p. 129]

$$h(t) = t^v e^{-at} \doteqdot \sqrt{v+1}\,(p+a)^{-v-1} = g(p),\ Re(v) > -1,\ Re(p+a) > 0$$

So with the help of Leibnitz's theorem, we get

$$h^n(t) \equiv e^{-at} \sum_{r=0}^{n} (-1)^{n-r} \frac{\sqrt{v+1}}{\sqrt{v-r+1}}\, {}^nC_r\, a^{n-r}\, t^{v-r}$$

Also we know that $h'(0) = h''(0) = \ldots\ldots = h^{n-1}(0) = 0$ if $v > n-1$

and $$t^{\lambda-\mu}\, h^n(t) \equiv e^{-at} \sum_{r=0}^{n} (-1)^{n-r} \frac{\sqrt{v+1}}{\sqrt{v-r+1}}\, {}^nC_r\, a^{n-r}\, t^{\lambda-\mu+v-r}$$

$$\frac{V}{\frac{1}{2}(1-\mu-\lambda),\ \frac{1}{2}(\mu-\lambda)} \sum_{r=0}^{n} \frac{(-1)^{n-r}\sqrt{v+1}\,\sqrt{\lambda-\mu+v-r+1}}{\sqrt{\lambda+v-r+1}}\, {}^nC_r\, p^{\mu-\lambda+r-v-1}\, a^{n-r}$$

$${}_2F_1\left[\begin{matrix}\lambda-\mu-r+v+1,\ v-r+1\\ \lambda-r+v+1\end{matrix}\ ;\ -\frac{a}{p}\right]$$

$Re(\lambda-\mu-r+v+1) > 0,\ Re(v-r+1) > 0,\ Re\ p > -Re\ a > 0,\ v > n-1.$

Substituting these values in the result (2·2), we get

$$\sum_{r=0}^{n} \frac{(-1)^{n-r}\sqrt{\lambda-\mu+v-r+1}}{\sqrt{\lambda-r+v+1}}\, {}^nC_r\, a^{n-r}\, p^{\mu-\lambda+r-v-1}$$

$${}_2F_1\left[\begin{matrix}\lambda-\mu-r+v+1,\ v-r+1\\ \lambda-r+v+1\end{matrix}\ ;\ -\frac{a}{p}\right] \stackrel{\omega}{\underset{\mu}{=}} t^{n-\lambda}\,(t+a)^{-v-1} \quad (3·1)$$

$Re(\lambda-\mu-r+v+1) > 0,\ Re(v-r+1) > 0,\ Re\ p > -Re\ a > 0,\ v > n-1$

with the help of the result (2, p. 201), (3·1) can be written as

$$\sum_{r=0}^{n} \frac{(-1)^{n-r}\sqrt{\lambda-\mu-r+v+1}}{\sqrt{\lambda-r+v+1}}\, {}^nC_r\, a^{n-r}\, p^{-\lambda+\mu+r-v-1}$$

$${}_2F_1\left[\begin{matrix}\lambda-\mu-r+v+1,\ v-r+1\\ \lambda-r+v+1\end{matrix}\ ;\ -\frac{a}{p}\right]$$

$$= \frac{p^{\mu-v-\lambda+n-1}\sqrt{\lambda-n-\mu+v+1}}{\sqrt{\lambda-n+v+1}}\ {}_2F_1\left[\begin{matrix}v+1,\ \lambda-n-\mu+v+1\\ \lambda-n+v+1\end{matrix}\ ;\ -\frac{a}{p}\right] \quad (3·2)$$

$Re(\lambda-\mu-r+v+1) > 0,\ Re(v-r+1) > 0,\ Re\ p > -Re\ a > 0,\ v > n-1.$

Example II

If we take (1, p. 195)

$$h(t) = I_v(at) \doteqdot a^v(p^2 - a^2)^{-\frac{1}{2}} (p + \sqrt{p^2-a^2})^{-v} = g(p)$$

$$Re\ p > Re\ a,\ Re(v) > -1$$

Differentiating $h(t)$, n^{th} times, we have

$$h^n(t) \equiv (\tfrac{1}{2})^n\ a^n\ {}^nC_r\ I_{v+2r-n}\ (at)$$

So that (1, p. 196)

$$t^{-\mu}\ h^n(t) \equiv (\tfrac{1}{2})^n \sum_{r=0}^{n} a^n\ {}^nC_r\ t^{-\mu}\ I_{v+2r-n}\,(at) \doteqdot a^n\ (\tfrac{1}{2})^n$$

$$\sum_{r=0}^{n} \sqrt{v+2r-n-\mu+1}\ {}^nC_r\ (p^2-a^2)^{\mu/2-\frac{1}{2}}\ P_{-\mu}^{-v-2r+n}\left(\frac{p}{\sqrt{p^2-a^2}}\right) = \phi(p)$$

$$Re\ p > Re\ a,\ Re(v-\mu+2r) > n-1$$

Since we know that $h'(0) = h''(0) = \ldots. = h^{n-1}(0) = 0$.

Substituting these values in the Corollary I, we get

$$\int_p^\infty t^n(t^2-a^2)^{-\frac{1}{2}}\ (t+\sqrt{t^2-a^2})^{-v}\ (t-p)^{\mu-1}\ dt = \Gamma\mu(\tfrac{1}{2})^n\ a^{n-v}$$

$$\sum_{r=0}^{n} \sqrt{v+2r-n-\mu+1}\ {}^nC_r\ (p^2-a^2)^{\mu/2-\frac{1}{2}}\ P_{-\mu}^{-v-2r+n}\left(\frac{p}{\sqrt{p^2-a^2}}\right) \quad (3\cdot3)$$

$$Re(v-\mu+2r) > n-1,\ Re(p) > Re\ a$$

On taking $n = 0$, (3·3) gives us

$$\int_p^\infty (t^2-a^2)^{-\frac{1}{2}}\ (t+\sqrt{t^2-a^2})^{-v}\ (t-p)^{\mu-1}\ dt$$

$$= \Gamma\mu\ \Gamma(v-\mu+1)\ a^{-v}\,(p^2-a^2)^{\mu/2-\frac{1}{2}}\ P_{-\mu}^{-v}\left(\frac{p}{\sqrt{p^2-a^2}}\right)$$

$$Re(v-\mu) > -1,\ Re\ p > Re\ a.$$

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REFERENCES

1. Erdelyi, A. Tables of Integral Transforms Vol. I *McGraw Hill, New York*, (1954).
7. Erdelyi, A Tables of Integral Transforms. Vol. II, *McGraw Hill, New York*, (1954).
3. Goldstein, S. Operational representation of Whittaker's confluent Hypergeometric function and Weber's parabolic cylinder function. *Prcc. Lond. Math. Soc.*, 34 (2) : 103–125, (1932).
4. Varma, R. S. *Current Science*, 15 : 16–17, (1947).

CLAY MINERALOGICAL STUDIES ON ALIGARH SOILS (PART—I)

By

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ABSTRACT

The mineralogical makeup of the six soil types of Aligarh district has been studied by X-ray diffraction and chemical analysis. Illite has been found to be the dominant constituent of all the clays studied. The other minerals detected are chlorite, labradorite and calcite. The studies reveal that all the soils are more or less similar in their mineral composition but their proportions vary.

The district of Aligarh, a trough-like depression, occupies an area of about 5000 sq. km. Its alluvial plane has an arid to subarid climate with an annual rainfall between 500 and 625 mm. A considerable part of its land has been affected by sodic salinization and is known as 'Usar' meaning infertile. The permeability of these soils is low due to an underlying bed of hard 'kankar'. Some of the lands are so barren that the average agricultural production of the district is low. According to a survey carried out by the Department of Agriculture, U. P.[1], six principal soil types have been recognised in the district, *viz.*, Ganga khadir (type I), Eastern uplands (type II), Central lowlands (type III), Western uplands (type IV), Yamuna khadir (type V) and Trans-Yamuna khadir (type VI). The classification is based on pedological principles but no systematic study has been made on the mineralogical nature of these different soil types. It was, therefore, thought worthwhile to study some of the mineralogical aspects of these soils.

EXPERIMENTAL

Soil samples (0–6″) were collected from the representative areas of the district as indicated in the soil map.[1] The representative nature of the samples was checked by the determination of mechanical composition, exchangeable cation and soluble salt contents. The mechanical analysis was done by the pipette method using sodium oxalate as the dispersing agent. The results are recorded in Table I. The pH and conductivity measurements were made with Beckman pH meter model G and Philips conductivity meter respectively in 1 : 5 soil extracts. Exchangeable sodium and potassium were estimated in the neutral ammonium acetate extract and calcium and magnesium in the barium chloride-triethanolamine extract[2] of the soils. The values for exchangeable calcium were further corrected according to Peech[3]. The results obtained are given in Table II.

Clays separated from the soils were subjected to analysis for silicon, aluminium and iron by the semi-microchemical method of Corey and Jackson[4] in which the elements were estimated by spectrophotometric technique following Na_2CO_3 fusion using molybdosilicic acid, aluminon and potassium thiocyanate respectively as colour reagents Total Ca plus Mg was determined in the solution (left after removal of Fe, Al and Si) by versene titration and Ca alone by ammonium oxalate precipitation followed by permanganate titration. Mg was calculated by difference. The results for Ca and Mg were checked by the method of Gysling and Schwarzenbach[5] using murexide and eriochrome black *T* as indicators following HF treatment. Sodium and potassium were estimated on a macroscale in HF treated samples with magnesium uranyl acetate following

Caley and Foulk[6] in case of sodium, and cobaltinitrite[7] procedure in case of potassium. Blanks were carried out as control for impurities of reagents and contamination from glassware. The results are summarized in Table III.

TABLE 1

Mechanical Analysis of Typical Aligarh Soils

Soil Type	Description	Coarse sand	Medium sand	Fine sand	Coarse silt	Medium silt	Fine silt	Clay
Type I	Ganga khadir, light grey, low permeability	–	6·1	48·0	13·20	18·50	8·10	6·06
Type II	Eastern uplands, light brown, fair permeability	–	3·9	67·21	4·00	4·90	10·82	8·80
Type III	Central lowlands, whitish grey, low permeability	1·3	2·0	57·52	3·60	11·00	10·70	13·80
Type IV	Western uplands, brownish, high permeability	–	2·8	71·09	5·13	7·00	5·40	8·60
Type V	Yamuna khadir, dark grey, low permeability	0·3	4·2	54·85	5·40	14·30	6·90	14·00
Tppe VI	Trans-Yamuna khadir, greyish, low permeability	–	0·9	69·60	3·00	9·30	6·40	10·90

TABLE 2

Physical and Chemical Characteristics of Typical Aligarh Soils

Soil type	pH	E.C. mmhos/cm at 25°C	Water soluble salts percentage	Exchangeable cations inclusive of soluble salts expressed as m.e/100 gm soil			
				Na	K	Ca	Mg
Type I	8·15	0·3209	0·103	0·91	4·59	4·02	2·38
Type II	8·00	0·1218	0·039	0·27	4·57	0·36	0·89
Type III	9·10	0·7740	0·248	8·73	2·34	4·18	0·61
Type IV	8·40	0·4873	0·156	4·15	2·34	0·66	0 66
Type V	8·30	0·6579	0·191	4·09	3·96	6·42	2·58
Type VI	8·00	0·1644	0·049	1·94	1·01	2·22	2·02

TABLE 3

Chemical Analysis of Clays Obtained from Typical Aligarh Soils
(Percentage on Oven Dry Basis)

Soil type	SiO_2	R_2O_3	Al_2O_3	Fe_2O_3	CaO	MgO	Na_2O	K_2O	Molar ratios SiO_2/R_2O_3	Molar ratios SiO_2/Al_2O_3
Type I	36·37	27·98	19·40	8·58	6·53	4·90	1·23	4·62	2·49	3·19
Type II	38·51	32·67	19·90	12·77	1·69	4·08	1·47	4·27	2·33	3·29
Type III	44·92	29·43	20·66	8·77	2·76	3·55	2·46	4·89	2·95	3·69
Type IV	42·78	30·65	18·64	12·01	2·31	3 59	1·50	3·70	2·76	3·90
Type V	43·85	28·87	19·15	9·72	5·29	4·28	1·85	4·38	2·94	3·89
Type VI	48·13	29·05	19·90	9·15	1·97	3·30	0·88	5·80	3·37	4·11

X-ray diffraction patterns of the six soils and their clays in sodium form, both before and after thermal treatments, were obtained using Nonious quadruple focussing camera with a built-in monochromator and Cu $K\alpha_1$ radiation (wave length = 1·5405 A°). The samples were ground and passed through a 200 mesh sieve and enclosed in a specimen holder with cellophane tape on either side. The X-ray tube was operated at 32 Kv and 8 mA. The exposure time was 25 hours. It was observed that the X-ray patterns given by the six soils were almost similar to each other. For economy of space, X-ray data for one soil and its clay fraction only are given in Table IV.

TABLE 4

Lattice spacing in A°-unit and estimated intensities of the lines in X-ray diffraction patterns of Aligarh soil types 3

Raw Soil $d$hkl	I*	Clay in Na form $d$hkl	I	Clay heated to 600°C $d$hkl	I	Clay mineral identified
–	–	14·00	VVW	14·00	VW	Cl
10·00	VVW	10·00	W	10·10	MS	I
–	–	7·20	VVW	7·20	VVW	Cl
–	–	–	–	5·00	VVW	I
4·50	VVW	4·50	S	4·50	S	I
4·26	S	4·26	MS	4·26	MS	Q
4·05	VW	–	–	4·05	VW	L
3·89	VVW	3·89	VVW	3·89	VVW	I
3·76	VVW	–	–	3·75	VVW	L
3·65	VVW	–	–	3·65	VVW	L,I
3·50	VVW	3·50	VVW	3·50	W	Cl
3·35	VS	3·35	VS	3·35	VS	I,Q
3·20	VW	3·20	VW	3·20	W	L
3·00	VVW	3·00	VVW	3·00	VVW	C
2·84	VVW	–	–	2·85	VVW	Cl,I
2·55–2·60	VW	2·55–2·60	S	2·55–2·60	S	I,Cl
2·45	W	2·45	VW	2·45	VW	Q,I
2·39	VVW	2·39	VVW	2·39	VVW	C,L,I
2·28	W	2·28	VW	2·28	VW	Q,C
2·23	VW	2·23	VW	2·23	VW	Q,I
2·12	VW	2·12	VW	2·12–2·14	VW	Q,I
2·00–1·98	VW	2·00–1·98	VW	2 00–1·98	VW	C,L,Q & C
1·82	MS	1·82	W	1·82	W	Q
1·67	W	1·67	VW	1·67	VW	Q
1·66	VW	1·66	VW	1 66	VW	Q,I
1 54	MS	1·53–1·54	VW	1·53–1·54	W	Q Cl
1·50	VW	1·50	VW	1·50	VVW	I
1·45	VVW	1·45	VVW	1·45	VVW	Q
1·38	M	1·37	VW	1·37	VW	Q

*Intensities are estimated in the following order :

VS (Very strong), S (Strong), MS (Medium strong), M (Medium) W (Weak), VW (Very weak) and VVW (Very very weak).

Illite, quartz, labradorite (felsper), chlorite and calcite are indicated by the letter I, Q, L, Cl and C respectively.

RESULTS AND DISCUSSION

An examination of the mechanical analysis data (Table I) reveals that the soils vary in texture from sandy loam to clayey loam. The pH of the soils increases with increasing exchangeable sodium content. The permeability and colour, the contents of soluble salts, exchangeable sodium, potassium, calcium and magnesium are in general agreement with the earlier publisihed data on these soils[1], excepting soil types I and V which showed some variation due to their immature nature and the possibility of annual inundations in these areas.

The total elemental and X-ray analysis of the clay fractions reveal that though the relative amounts may somewhat vary, the same groups of minerals are present in all the soils. The clays are characterised by high silica-alumina (3·19 to 4·11) and silica-sesquioxide ratios (2·33 to 3·77), which along with their high potash contents[8] (3·70 to 5·80) are indicative of the presence of illite in all the samples. Further, the presence of magnesia[9] suggests the existence of chlorite and/or vermiculite in them.

X-ray patterns confirm the presence of illite as the dominant mineral and of chlorite in smaller quantities. Lines at 10·00, 4·50, 3·89 and 3·35 A° indicate the existence of illite and at 14·00, 7·20, 3·50 and 2·84 A° indicate chlorite. Because of the presence of chlorite in very small quantities in the soils, the 14 A° line for it could not be detected in the X-ray patterns of soils, but it could be clearly observed in the clay fractions in which the chlorite content was high. The presence of chlorite was also confirmed by taking X-ray patterns of clays heated to 600°C for half an hour, as a result of which the intensity of 14 A° was increased. On the basis of intensities of X-ray lines it can be suggested that soil types I, III, V and VI are somewhat richer in illite than II and IV and the quantity of chlorite in soil types I and V is slightly greater than in types II, III, IV and VI. Strong lines at 4·26, 3·35 and 1·82 A° indicate the presence of quartz. 4·05, 3·75, 3·65 and 3·20 A° lines correspond to those for labradorite. A greater quantity of quartz is present in soil types III, IV and V than in II and VI. Calcite is also present in these soils as revealed by lines at 3·00, 2·28 and 2·09 A° and the CaO content of the clays. Kaolinite, montmorillonite and vermiculite could not be detected either in the soils or their clay fractions. Ferruginous minerals and some amorphous materials are also indicated in all samples. Further work along these lines is in progress and will be reported later.

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REFERENCES

1. Department of Agriculture, U. P. *Soil Survey and Soil Work in U. P. Vols.* I & II, (1950)
2. Mehlich, A. *J. Assoc. of Ag. Chem.*, **36** : 445, (1953).
3. Peech, M et al. *U. S. D A. Cir* 757, (1947).
4. Corey, R. B. and Jackson, M. L *Anal. Chem.*, **25** : 624 (1953).
5. Gysling, H. and Schwarzenbach, G. *Helv Chem. Acta.*, **32** : 1484, (1949).
6. Caley. E R. and Foulk, C. W. *J. Am Chem. Soc.*, **51** : 1664, (1929).
7. Jackson, M L. Soil Chemical Analysis. *Constable & Co. Ltd.*, (1958).
8. Ross, C. S. and Hendricks, S. B. *Geol. Survey Paper* 250-B, *U S. Dept. of the Interior*, (1945).
9. Grim, R. E. Clay mineralogy *McGraw Hill Book Co Inc.*, (1953).

PROPAGATION OF CURVED SHOCKS IN PSEUDO-STATIONARY THREE-DIMENSIONAL CONDUCTING GAS FLOWS

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ABSTRACT

For non-dissipative flows, the first and second order partial derivatives of flow and field variables have been obtained and consequently the curvature and torsion of a streak-line behind a three dimensional pseudo-stationary curved magnetogasdyanamic shock wave have been determined. Furthermore the third order derivatives of the fluid velocity U_i relative to the shock wave have been determined and consequently the curvature and torsion of a vortex line just behind the shock surface have been calculated. These results have been obtained by making use of certain matrices and it has been assumed that the flow and field ahead of the shock are uniform.

§ 1. INTRODUCTION

The basic equations governing the unsteady motion of a conducting gas with dissipative mechanisms, such as viscosity, thermal conductivity and electrical resistance absent, are

$$\frac{dH_i}{dt} - H_j \frac{\partial u_i}{\partial x_j} + H_i \frac{\partial u_j}{\partial x_j} = 0, \tag{1·1}$$

$$\frac{\partial H_i}{\partial x_i} = 0, \tag{1·2}$$

$$\frac{d\rho}{dt} + \rho \frac{\partial u_i}{\partial x_i} = 0, \tag{1·3}$$

$$\rho \frac{du_i}{dt} + \frac{\partial p^*}{\partial x_i} - \frac{1}{4\pi} H_j \frac{\partial H_i}{\partial x_j} = 0, \tag{1·4}$$

where $p^* = p + H^2/8\pi$,

$$\frac{d\eta}{dt} = 0, \tag{1·5}$$

$$p = \exp\left(\frac{\eta}{JC_v}\right)\rho^\gamma. \tag{1·6}$$

In the above equations, H_i denotes the components of the magnetic field vector, u_i the components of velocity vector, p the presure, ρ the density, η the specific entropy, J the mechanical equivalent of heat and γ is the ratio of two specific heats C_p and C_v assumed constant. We assume the motion referred to a system of rectangular co-ordinates x_i. Furthermore, it is to be understood in the above and in the following discussion, unless the contrary is stated, that an index which occurs twice in a term is to be summed over the admissible values of the index. Since there is no distinction between covarient and contravarient indices within a rectangular system, we may write an index as a subscript or a superscript without modifying the value of the term in which the index occurs.

The shock configuration in a three-dimensional gas flow at time may be represented by the equations

$$x_i = ta_i\,(y^1, y^2), \tag{1·7}$$

where y^1 and y^2 are the Gaussian co-ordinates on the shock surface. This substitution leads to the following form of the differential operators in the equations (1·1) to (1·5):

$$\frac{df}{dt} = \frac{1}{t}(u_i - a_i) f_{,i} = \frac{1}{t} U_i f_{,i} \tag{1·8}$$

$$\frac{\partial f}{\partial x_i} = \frac{1}{t} f_{,i} \tag{1·9}$$

where $f(x_1/t, x_2/t, x_3/t)$ is any function into which the co-ordinates and time enter in the manner indicated and the derivative $\partial f/\partial \eta_i$ has been written $f_{,i}$. The symbol U_i is defined in the equation (1·8).

When this substitution is introduced into the equations (1·1) to (1·5), they become

$$U_j H_{i,j} - H_j U_{i,j} + H_i U_{k,k} + 2H_i = 0, \tag{1·10}$$

$$H_{i,i} = 0, \tag{1·11}$$

$$\rho U_{i,i} + U_i \rho_{,i} + 3\rho = 0, \tag{1·12}$$

$$\rho U_i + \rho U_j U_{i,j} + p^*_{,i} - \frac{1}{4\pi} H_j H_{i,j} = 0, \tag{1·13}$$

$$U_i \eta_{,i} = 0. \tag{1·14}$$

A flow which meets these requirements is called pseudo-stationary.[1] From (1·7) we notice that the components of the velocity of the shock are given by

$$v_i = a_i. \tag{1·15}$$

Since the u_i are the components of the particle velocity, the quantities $U_i = u_i - a_i$ when evaluated on either side of the shock front, give the velocity of the flow relative to the shock at the corresponding points.

If a quantity f is evaluated on the upstream side of the shock surface, we shall denote it by f_1. Similarly if the quantity is evaluated downstream, we shall denote it by f. The jump of the quantity across the shock surface is expressed by

$$[f] \equiv f - f_1. \tag{1·16}$$

The expressions for the flow and field quantities in the region behind the shock surface are expressed in terms of their values infront of the shock surface by the relations[2].

$$[H_i] = S_H (H_{1i} - H_{1n} X_i), \tag{1·17}$$

$$[U_i] = \frac{U_{1n}}{H_{1n}} A_1 S_H H_{1i} - U_{1n} \frac{(1 + A_1 S_H)}{1 + S_H} S_H X_i, \tag{1·18}$$

$$[p^*] = S_H \frac{(1 - A_1)}{1 + S_H} \rho_1 U_{1n}^2, \tag{1·19}$$

$$[\rho] = \frac{\rho_1 S_H (1 - A_1)}{1 + A_1 S_H}, \tag{1·20}$$

where S_H is the magnetic field strength of the shock defined as

$$H_{1\alpha} S_H = [H_\alpha], \tag{1·21}$$

and

$$A_1 = \frac{H_{1n}^2}{4\pi\rho_1 U_{1n}^2} < 1. \tag{1·22}$$

For a perfect gas S_H is given by the relation[2]

$$C_1^2\,(A_1-1) = \frac{(A_1-1)\,U_{1n}^2}{2\,(1+S_H)}(2 + A_1S_H + \gamma A_1 S_H - \gamma S_H + S_H)$$

$$+\ \frac{A_1\,U^2_{1n}}{2\,H^2_{1n}}\ H_{1\alpha}\,H^{\alpha}{}_1\ (2 + A_1\,S_H + 2S_H + A_1\,S_H{}^2 + \gamma A_1\,S_H - \gamma\,S_H) \qquad (1{\cdot}23)$$

where

$$C_1{}^2 = \gamma\,p_1/\rho_1,$$

and

$$H_{1n} = H_{1i}\,X_i \quad ; \quad H_{1\alpha} = H_1{}^{\beta}\,a_{\alpha\beta} = H_{1i}\,X^i{}_{\alpha}\,;$$

X_i denoting the components of the unit vector normal to the shock surface and $X^i{}_{\alpha}$ are the components of a vector tangential to the shock surface.

§ 2. DERIVATION OF THE FIRST ORDER PARTIAL DERIVATIVES OF THE FLOW AND FIELD VARIABLES BEHIND THE SHOCK SURFACE

In the following discussion we take lines of curvatures as the Gaussian co-ordinate curves on the shock surface. We use Latin letters for the indices referring to the space variables and Greek letters for the indices referring to the surface variables. Thus the Latin indices range from 1 to 3 while Greek indices take the values 1 and 2.

The surface unit tangent vectors to the co-ordinate curves are $\delta_1{}^{\alpha}/\sqrt{a_{11}}$ and $\delta^{\alpha}{}_2/\sqrt{a_{22}}$ where $a_{\alpha\beta}$ are the components of the first fundamental form of the surface. The corresponding space components of the unit tangent vectors are $x^i{}_1/\sqrt{a_{11}}$ and $x^i{}_2/\sqrt{a_{22}}$ respectively, where we have put

$$x^i{}_{\alpha} \equiv \frac{\partial x_i}{\partial y^{\alpha}} = t\ \frac{\partial n_i}{\partial y^{\alpha}}. \qquad (2{\cdot}1)$$

Now differentiating the relations (1·17) to (1·20) with respect to y^1 and y^2, we get

$$U_{i,j}\,x^j{}_{\alpha/t} = U_{1i,j}\,x^j{}_{\alpha/t} + \overline{A}_{i\alpha/t} = A^*{}_{i\alpha/t}, \qquad (2{\cdot}2)$$

$$p^*{}_{,j}\,x^j{}_{\alpha/t} = p^*{}_{1,j}\,x^j{}_{\alpha/t} + \overline{B}_{\alpha/t} = B^*{}_{\alpha/t}, \qquad (2{\cdot}3)$$

$$\rho_{,j}\,x^j{}_{\alpha/t} = \rho_{1,j}\,x^j{}_{\alpha/t} + \overline{C}_{\alpha/t}\,C^*{}_{\alpha/t}, \qquad (2{\cdot}4)$$

$$H_{i,j}\,x^j{}_{\alpha/t} = H_{1i,j}\,x^j{}_{\alpha/t} + \overline{D}_{i\alpha/t} = D^*{}_{i\alpha/t}. \qquad (2{\cdot}5)$$

The explicit values of $\overline{A}_{i\alpha}$, $\overline{B}_{\alpha}$, $\overline{C}_{\alpha}$ and $\overline{D}_{i\alpha}$ are obtained by differentiating the right members of the equations (1·18), (1·19), (1·20) and (1·17) and are given in the subsequent work.

Now we set

$$C_{i\alpha} = x^i{}_{\alpha},\ \ C_{i3} = U_i. \qquad (2{\cdot}6)$$

If det $C_{ij} \neq 0$, then we can define D_{ij} by the relations

$$D_{ij} = \frac{\text{Cofactor of } C_{ji} \text{ in } \|\,C_{pq}\,\|}{|\,C_{pq}\,|},$$

$$D_{ik}\,C_{kj} = \delta_{ij},\ \ D_{ki}\,C_{jk} = \delta_{ij} \qquad (2{\cdot}7)$$

Further, define the quantities B_{ij} by

$$B_{ij} = U_{l,m}\,C_{li}\,C_{mj}. \qquad (2{\cdot}8)$$

By (2·7) we may invert this relation and obtain

$$U_{l,m} = B_{ij}\, D_{il}\, D_{jm}\,. \tag{2·9}$$

With the help of (2·2) and the relation

$$U_i = U_n\, X_i + U^{\alpha}\, x^i_{\alpha}, \tag{2·10}$$

we obtain

$$\begin{aligned} B_{\alpha\beta} &= A^*_{i\beta}\, x^i_{\alpha}, \\ B_{\alpha 3} &= x^i_{\alpha}\, A^*_{i\beta}\, U^{\beta} + U_n\, U_{i,j}\, x^i_{\alpha}\, X_j. \\ B_{3\alpha} &= A^*_{i\alpha}\, U_i, \\ B_{33} &= U^{\beta}\, A^*_{i\beta}\, U_i + U^2_n\, U_{i,j}\, X_i\, X_j + U_n\, U^{\alpha}\, U_{i,j}\, x^i_{\alpha}\, X_j. \end{aligned} \tag{2·11}$$

Furthermore, define the quantities $\bar{B}_{ij}$ by

$$\bar{B}_{ij} = H_{l,m}\, C_{li}\, C_{mj}, \tag{2·12}$$

which, when inverted, gives

$$H_{l,m} = \bar{B}_{ij}\, D_{il}\, D_{jm}, \tag{2·13}$$

By virtue of (2·5) and (2·10), the relations (2·12) give

$$\begin{aligned} \bar{B}_{\alpha\beta} &= D^*_{i\beta}\, x^i_{\alpha}, \\ \bar{B}_{\alpha 3} &= x^i_{\alpha}\, D^*_{i\beta}\, U^{\beta} + U_n\, H_{i,j}\, x^i_{\alpha}\, X_j, \\ \bar{B}_{3\alpha} &= D^*_{i\alpha}\, U_i, \\ \bar{B}_{33} &= U^{\beta}\, D^*_{i\beta}\, U_i + U^2_n\, H_{i,j}\, X_i\, X_j + U_n\, U^{\alpha}\, H_{i,j}\, x^i_{\alpha}\, X_j. \end{aligned} \tag{2·14}$$

The values of $U_{i,j}\, x^i_{\alpha}\, X_j$, $U_{i,j}\, X_i\, X_j$, $H_{i,j}\, x^i_{\alpha}\, X_j$ and $H_{i,j}\, X_i\, X_j$ can be obtained from the basic equations in the following way.

With the help of (2·2), (2·5), (2·10) and the relation

$$H_i = H_n\, X_i + H^{\alpha}\, x^i_{\alpha}, \tag{2·15}$$

the equation (1·10) gives

$$\begin{aligned} 2H_i + U_n\, X_j\, H_{i,j} + U^{\beta}\, D^*_{i\beta} - H_n\, X_j\, U_{i,j} \\ - H^{\beta}\, A^*_{i\beta} + H_i\, U_{k,k} = 0. \end{aligned} \tag{2·16}$$

Multiplying (1·13) by x^i_{α} and using (2·2), (2·3), (2·5), (2·10), (2·15) we obtain

$$\begin{aligned} \rho U_{\alpha} + \rho U^{\beta}\, A^*_{i\beta}\, x^i_{\alpha} + \rho U_n\, U_{i,j}\, x^i_{\alpha}\, X_j + B^*_{\alpha} \\ - \frac{1}{4\pi}\, x^i_{\alpha}\, D^*_{i\beta}\, H^{\beta} - \frac{1}{4\pi}\, H_n\, H_{i,j}\, x^i_{\alpha}\, X_j = 0. \end{aligned} \tag{2·17}$$

Multiplying (2·16) by x^i_{α} we get

$$\begin{aligned} 2H_{\alpha} + U_n\, H_{i,j}\, x^i_{\alpha}\, X_j + U^{\beta}\, D^*_{i\beta}\, x^i_{\alpha} - H_n\, U_{i,j}\, x^i_{\alpha}\, X_j \\ - H^{\beta}\, A^*_{i\beta}\, x^i_{\alpha} + H_{\alpha}\, U_{k,k} = 0. \end{aligned} \tag{2·18}$$

Multiplying (2·17) by U_n and (2·18) by $\dfrac{H_n}{4\pi}$ and adding we get

$$U_{i,j}\, x^i_{\alpha}\, X_j = L_{\alpha} + \frac{H_n}{R}\, H_{\alpha}\, U_{k,k}, \tag{2·19}$$

where

$$\begin{aligned} L_{\alpha} = \frac{4\pi}{R}\Big(& U_n\, B^*_{\alpha} + \rho U_n\, x^i_{\alpha}\, A^*_{i\beta}\, U^{\beta} - \frac{1}{4\pi}\, U_n\, x^i_{\alpha}\, D^*_{i\beta}\, H^{\beta} \\ & + \frac{H_n}{4\pi}\, x^i_{\alpha}\, D^*_{i\beta}\, U^{\beta} - \frac{H_n}{4\pi}\, x^i_{\alpha}\, A^*_{i\beta}\, H^{\beta} + \rho U_n\, U_{\alpha} + \frac{H_n\, H_{\alpha}}{2\pi} \Big) \end{aligned}$$

and $$R = H^2{}_n - 4\pi \rho U^2{}_n.$$

Furthermore, multiplying (2·17) by H_n and (2·18) by ρU_n and adding we obtain

$$H_{i \cdot j}\, x^i{}_\alpha X_j = M_\alpha + \frac{4\pi\rho}{R} U_n H_\alpha U_{k,k} \tag{2·20}$$

where $$M_\alpha = \frac{4\pi}{R}\Big(H_n B^*{}_\alpha + \rho H_n U^\beta A^*{}_{i\beta}\, x^i{}_\alpha - \frac{1}{4\pi} H_n x^i{}_\alpha D^*{}_{i\beta} H^\beta$$
$$+ \rho U_n x^i{}_\alpha D^*{}_{i\beta} U^\beta - \rho U_n x^i{}_\alpha A^*{}_{i\beta} H^\beta + \rho H_n U_\alpha + 2\rho U_n H_\alpha\Big).$$

Now differentiating (1·6) with respect to a_i and making use of (1·14) and (1·12) we obtain

$$p_{,i}\, U_i = -\gamma p (U_{k,k} + 3). \tag{2·21}$$

Further, multiplying (1·13) by U_i and using (2·21), (2·2), (2·5), (2·10), (2·15) we obtain

$$\rho U^2{}_i + \rho U^\beta A^*{}_{i\beta} U_i - \gamma p U_{k,k} - 3\gamma p + \frac{1}{4\pi} H_j A^*{}_{j\alpha} U^\alpha$$
$$- \frac{1}{4\pi} H^\alpha D^*{}_{i\alpha} U_i + \rho U^2{}_n U_{i,j} X_i X_j$$
$$+ \rho U_n U^\alpha U_{i,j}\, x^i{}_\alpha X_j + \frac{1}{4\pi}(U_n H^\alpha - H_n U^\alpha) H_{i,j}\, x^i{}_\alpha X_j = 0. \tag{2·22}$$

In this equation, substituting the value of $U_{i,j}\, x^i{}_\alpha X_j$ from (2·19) and $H_{i \cdot j}\, x^i{}_\alpha X_j$ from (2·20) we obtain

$$U_{i,j} X_i X_j = N + N' U_{k,k}, \tag{2·23}$$

where $$N = -\frac{1}{\rho U^2{}_n}\Big\{ \rho U^2{}_i + \rho U^\beta A^*{}_{i\beta} U_i + \frac{1}{4\pi} H_j A^*{}_{j\alpha} U^\alpha$$
$$- \frac{1}{4\pi} U_i D^*{}_{i\alpha} H^\alpha - 3\gamma p + \rho U_n U^\alpha L_\alpha + \frac{1}{4\pi} M_\alpha (U_n H^\alpha - H_n U^\alpha)\Big\},$$

and

$$N' = -\frac{1}{\rho U^2{}_n}\Big\{ -\gamma p + \rho U_n U^\alpha \frac{H_n}{R} H_\alpha + (U_n H^\alpha - H_n U^\alpha)\rho U_n \frac{H_\alpha}{R}\Big\}.$$

Now if we multiply (2·16) by X_i and use (2·23) we obtain

$$H_{i,j} X_i X_j = -\frac{1}{U_n}(2H_n + U^\beta D^*{}_{i\beta} X_i - N H_n - H^\beta A^*{}_{i\beta} X_i)$$
$$+ \frac{H_n}{U_n}(N' - 1) U_{k,k}. \tag{2·24}$$

To find the value of $U_{k,k}$ we have from the relation (2·9):

$$U_{k,k} = (B_{\alpha\beta} D_{\alpha k} D_{\beta k} + B_{3\beta} D_{3k} D_{\beta k}) + (B_{\alpha 3} D_{\alpha k} D_{3k} + B_{33} D_{3k} D_{3k}), \tag{2·25}$$

which with the help of (2·11), (2·19) and (2·23) gives the value of $U_{k,k}$.

To find the pressure gradient we have from (1·13) the relation

$$p^*{}_{,i} = \frac{1}{4\pi} H_j H_{i,j} - \rho U_j U_{i,j} - \rho U_i,$$

which with the help of (2·9), (2·13), (2·6) and (2·7) gives

$$p^*{}_{,i} = -\rho B_{l3} D_{li} - \rho U_i + \frac{1}{4\pi} H_j \bar{B}_{lm} D_{li} D_{mj}. \tag{2·26}$$

To evaluate the density gradient we define the quantities d_i by

$$d_i = \rho_{,j}\ C_{ji}, \tag{2·27}$$

which, when inverted, gives

$$\rho_{,m} = d_i\ D_{im}. \tag{2·28}$$

The relations (2·27), in consequence of (2·4), give

$$d_\alpha = C^*{}_\alpha. \tag{2·29}$$

Also

$$d_3 = \rho_{,j}\ U_j = -\rho(U_{k,k}+3). \tag{2·30}$$

where we have used the equation (1·12).

To find the value of $A^*{}_{i\alpha}$, $B^*{}_\alpha$, $C^*{}_\alpha$, $D^*{}_{i\alpha}$ we assume that the flow and field ahead of the shock are uniform. Then we have from the jump conditions (1·17) to (1·20):

$$\begin{aligned} D^*{}_{i\alpha/t} &= \overline{D}_{i\alpha/t} = \partial H_i/\partial y^\alpha \\ &= S_H\,(H_{1\gamma}\,X_i + H_{1n}\,x^i{}_\gamma)\,a^{\beta\gamma}\,b_{\beta\alpha} + H_1{}^\beta\,x^i{}_\beta\,\frac{\partial S_H}{\partial y^\alpha}, \end{aligned} \tag{2·31}$$

$$\begin{aligned} \frac{A^*{}_{i\alpha}}{t} = \frac{\overline{A}_{i\alpha}}{t} - \frac{x^i{}_\alpha}{t} = \frac{\partial U_i}{\partial y^\alpha} &= \frac{\partial S_H}{\partial y^\alpha}\left(\frac{U_{1n}}{H_{1n}}A_1 H_1{}^\beta\,x^i{}_\beta - \frac{U_{1n}\,(1-A_1)}{(1+S_H)^2}X_i\right) \\ &+ \frac{\partial H_{1n}}{\partial y^\alpha}\left(\frac{S_H}{4\pi\rho_1 U_{1n}}\,H_1{}^\beta\,x^i{}_\beta + \frac{S_H\,H_{1n}}{4\pi\rho_1 U_{1n}}\,\frac{(1-S_H)}{(1+S_H)}\,X_i\right) \\ &+ \frac{\partial U_{1n}}{\partial y^\alpha}\left(-\frac{A_1}{H_{1n}}S_H\,H_1{}^\beta\,x^i{}_\beta - \frac{S_H}{1+S_H}\,(1+A_1)\,X_i\right) \\ &+ \frac{\partial X}{\partial y^\alpha}(-S_H\,U_n) - \frac{x^i{}_\alpha}{t}, \end{aligned} \tag{2·32}$$

$$\begin{aligned} \frac{B^*{}_\alpha}{t} = \frac{\overline{B}_\alpha}{t} = \frac{\partial p^*}{\partial y^\alpha} &= \rho_1 U_{1n}^2\,\frac{(1-A_1)}{(1+S_H)^2}\,\frac{\partial S_H}{\partial y^\alpha} + 2\rho_1\,U_{1n}\,\frac{S_H}{1+S_H}\,\frac{\partial U_{1n}}{\partial y^\alpha} \\ &- \frac{H_{1n}}{2\pi}\cdot\frac{S_H}{1+S_H}\,\frac{\partial H_{1n}}{\partial y^\alpha}, \end{aligned} \tag{2·33}$$

$$\begin{aligned} \frac{C^*{}_\alpha}{t} = \frac{\overline{C}_\alpha}{t} = \frac{\partial \rho}{\partial y^\alpha} &= \rho_1\,\frac{(1-A_1)}{(1+A_1\,S_H)^2}\,\frac{\partial S_H}{\partial y^\alpha} \\ &- \frac{2A_1\,S_H\,(1+S_H)}{(1+A_1\,S_H)^2}\left(\frac{1}{H_{1n}}\,\frac{\partial H_{1n}}{\partial y^\alpha} - \frac{1}{U_{1n}}\,\frac{\partial U_{1n}}{\partial y^\alpha}\right), \end{aligned} \tag{2·34}$$

where $\dfrac{\partial S_H}{\partial y^\alpha}$ is given from (1·23) as

$$\begin{aligned} &\frac{\partial S_H}{\partial y^\alpha}\Big\{2\,C_1^2\,(A_1-1) - (A_1-1)\,(A_1+\gamma\,A_1-\gamma+1)\,U_{1n}^2 \\ &- (3A_1\,S^2{}_H + (4A_1+4+2\gamma\,A_1-2\gamma)\,S_H + A_1+\gamma A_1-\gamma+4)\,\frac{H_{1\delta}\,H_1{}^\delta}{4\,\pi\rho_1}\Big\} \\ &= \frac{H_{1n}}{2\pi\rho_1}\,\frac{\partial H_{1n}}{\partial y^\alpha}\Big\{\frac{1}{4\,\pi\,\rho_1\,U_{1n}^2}\,S_H\,(1+S_H)\,(\gamma+1+S_H) \\ &\times H_{1\delta}\,H_1{}^\delta - \frac{2C_1^2}{U_{1n}^2}\,(1+S_H) - A_1\,S_H{}^3 + S_H{}^2\,(\gamma-2-2A_1-\gamma A_1) \\ &+ S_H(\gamma\,A_1+A_1-4-\gamma)\Big\}. \end{aligned} \tag{2·35}$$

The values of $\frac{\partial X_i}{\partial y^\alpha}$, $\frac{\partial U_{1n}}{\partial y^\alpha}$ and $\frac{\partial H_{1n}}{\partial y^\alpha}$ are given by the relations

$$\frac{\partial X_i}{\partial y^\alpha} = - a^{\beta\gamma}\, b_{\beta\alpha}\, x^i{}_\gamma, \tag{2·36}$$

$$\frac{\partial U_{1n}}{\partial y^\alpha} = - U_{1\gamma}\, a^{\beta\gamma}\, b_{\beta\alpha} - \frac{\partial v}{\partial y^\alpha}, \tag{2·37}$$

$$\frac{\partial H_{1n}}{\partial y^\alpha} = - H_{1\gamma}\, a^{\beta\gamma}\, b_{\beta\alpha}. \tag{2·38}$$

where v denotes the speed of the shock in the normal direction.

§ 3. CALCULATION OF SECOND ORDER DERIVATIVES AND TORSION OF A STREAK-LINE BEHIND THE SHOCK SURFACE

Let the second order derivatives of the flow and field variables be given by the relations

$$U_{l,mn} = I_{ijk}\, D_{il}\, D_{jm}\, D_{kn}, \tag{3·1}$$

$$p^*{}_{,mn} = J_{jk}\, D_{jm}\, D_{kn}, \tag{3·2}$$

$$\rho_{,mn} = K_{jk}\, D_{jm}\, D_{kn}, \tag{3·3}$$

$$H_{l,mn} = \bar{I}_{ijk}\, D_{il}\, D_{jm}\, D_{kn}, \tag{3·4}$$

where

$$I_{ijk} = U_{l,mn}\, C_{li}\, C_{mj}\, C_{nk}, \text{ (symmetric in } j \text{ and } k) \tag{3·5}$$

$$J_{jk} = p^*{}_{,mn}\, C_{mj}\, C_{nk}, \text{ (symmetric in } j \text{ and } k) \tag{3·6}$$

$$K_{jk} = \rho_{,mn}\, C_{mj}\, C_{nk}, \text{ (symmetric in } j \text{ and } k) \tag{3·7}$$

$$\bar{I}_{ijk} = H_{l,mn}\, C_{li}\, C_{mj}\, C_{nk}, \text{ (symmetric in } j \text{ and } k) \tag{3·8}$$

Now differentiating the relations (2·9), (2·26), (2·28) and (2·13) with respect to y^α we obtain

$$U_{l,mn}\, x^n{}_\alpha = P_{lm\alpha}, \tag{3·9}$$

$$p^*{}_{,mn}\, x^n{}_\alpha = Q_{m\alpha}, \tag{3·10}$$

$$\rho_{,mn}\, x^n{}_\alpha = R_{m\alpha}, \tag{3·11}$$

$$H_{l,mn}\, x^n{}_\alpha = \bar{P}_{lm\alpha}, \tag{3·12}$$

where

$$P_{lm\alpha} = t\, \frac{\partial}{\partial y^\alpha}\left(B_{ij}\, D_{il}\, D_{jm}\right),$$

$$Q_{m\alpha} = t\, \frac{\partial}{\partial y^\alpha}\left(- \rho B_{l3}\, D_{lm} - \rho U_m + \frac{1}{4\pi} H_j\, \bar{B}_{lr}\, D_{lm}\, D_{rj} \right),$$

$$R_{m\alpha} = t\, \frac{\partial}{\partial y^\alpha}\left(d_i\, D_{im}\right),$$

$$\bar{P}_{lm\alpha} = t\, \frac{\partial}{\partial y^\alpha}\left(\bar{B}_{ij}\, D_{il}\, D_{jm}\right).$$

Thus we see that the quantities $P_{lm\alpha}$, $Q_{m\alpha}$, $R_{m\alpha}$ and $\bar{P}_{lm\alpha}$ are known in principle as they can be expressed in terms of the flow and field variables infront of the shock, the projection tensor $x^i{}_\alpha$, the surface normal X_i, two principal curvatures and their partial derivatives along the shock surface.

Now, by virtue of (3·9), (3·10), (3·11) and (3·12) the relations (3·5) to (3·8) give

$$I_{ij\alpha} = P_{lm\alpha}\, C_{li}\, C_{mj}, \tag{3·13}$$

$$J_{j\alpha} = Q_{m\alpha}\, C_{mj}, \tag{3·14}$$
$$K_{j\alpha} = R_{m\alpha}\, C_{mj}. \tag{3·15}$$
$$\bar{I}_{ij\alpha} = \bar{P}_{lm\alpha}\, C_{li}\, C_{mj}. \tag{3·16}$$

Hence the quantities $I_{ij\alpha}, J_{j\alpha}$, $K_{j\alpha}$ and $\bar{I}_{ij\alpha}$ are known in principle. The symmetry conditions satisfied by these quantities reduce this problem to the determination of I_{i33}, J_{33}, K_{33} and $\bar{I}_{i33}$ only. To find these we proceed as follows :

Differentiating the equations (1·10), (1·12) and (1·13) with respect to a_k we obtain respectively

$$U_{j,k}\, H_{i,j} - H_{j,k}\, U_{i,j} + H_{,k}\, U_{r,r} + 2\, H_{i,k} + U_j\, H_{i,jk} - H_j U_{i,jk} + H_i\, U_{r,rk} = 0. \tag{3·17}$$
$$\rho_{,k}\, U_{j,j} + \rho\, U_{j,jk} + U_{i,k}\, \rho_{,i} + U_i \rho_{,ik} + 3\rho_{,k} = 0, \tag{3·18}$$
$$\rho_{,k}\, U_i + \rho\, U_{i,k} + \rho_{,k}\, U_j\, U_{i,j} + \rho\, U_{j,k}\, U_{i,j} - \frac{1}{4\pi} H_{j,k}\, H_{i,j} + \rho\, U_j\, U_{i,jk} + p^*_{,ik} - \frac{1}{4\pi}\, H_j\, H_{i,jk} = 0. \tag{3·19}$$

Multiplying this equation by $C_{il}\, C_{km}$ and using the previous relations we obtain

$$X_{lm} + \rho\, I_{lm3} + J_{lm} - \frac{1}{4\pi} H^\beta\, \bar{I}_{lm\beta} - \frac{1}{4\pi} H_n\, X_j\, H_{i,jk}\, C_{il}\, C_{km} = 0, \tag{3·20}$$

where
$$X_{lm} = C_{il}\, C_{km}\, (\rho_{,k}\, U_i + \rho U_{i,k} + \rho_{,k}\, U_j\, U_{i,j} + \rho\, U_{j,k}\, U_{i,j} - \frac{1}{4\pi}\, H_{j,k}\, H_{i,j}).$$

Putting $l = m = 3$ in (3·20) and using (2·10) we get

$$X_{33} + \rho\, I_{333} + J_{33} - \frac{1}{4\pi}\, H^\beta\, \bar{I}_{33\beta} - \frac{1}{4\pi} H_n\;\; U_n\;\; H_{i,jk}\, X_i\, X_j\, U_k - \frac{1}{4\pi}\, H_n\, U^\alpha\, H_{i,jk}\, x^i{}_\alpha\, X_j\, U_k = 0. \tag{3·20a}$$

Further multiplying (3·17) by $U_i\, U_k$ and using the previous relations we obtain

$$Y + \bar{I}_{333} + H_i\, U_i\, U_k\, U_{r,rk} - I_{33\alpha}\, H^\alpha - H_n\, U_n\, U_{i,jk}\, X_i\, X_j\, U_k - H_n\, U^\alpha\, U_{i,jk}\, x^i{}_\alpha\, X_j\, U_k = 0, \tag{3·21}$$

where

$$Y = U_i\, U_k\, (H_{j,k}\, H_{i,j} - H_{j,k}\, U_{i,j} + H_{i,k}\;\; U_{r,r} + 2\, H_{i,k}) .$$

Furthermore differentiating the relation

$$p^* = p + H^2/8\pi$$

and eliminating $p_{,i}\; U_i$ with the help of the relation

$$p_{,i}\, U_i = \frac{\gamma p}{\rho}\, \rho_{,i}\, U_i$$

we obtain

$$-\frac{1}{4\pi}\;\; \rho\, H_j\, H_{j,i}\, U_i + \rho\, p^*_{,i}\, U_i - \gamma\, p\, \rho_{,i}\, U_i = 0.$$

This equation, in consequence of (1·12), gives

$$\gamma p\, U_{r,r} + 3\gamma p - \frac{1}{4\pi}\;\; H_j\, U_i\, H_{j,i} + p^*_{,i}\, U_i = 0.$$

Differentiating this relation with respect to a_k we obtain

$$\gamma p\, U_{r,rk} - \frac{1}{4\pi} U_i H_j H_{j,ik} + U_{i,}p^*{}_{,ik} + Z_k = 0, \tag{3·22}$$

where

$$Z_k = \gamma p_{,k} U_{r,r} + 3\, \gamma p_{,k} - \frac{1}{4\pi} (H_j U_i)_{,k} H_{j,i} + U_{i\cdot k}\, p^*{}_{,i}.$$

Multiplying (3·22) by U_k, using (2·10) and (2·15) we obtain

$$J_{33} + \gamma p\, U_k U_{r,rk} + Z_k U_k - \frac{1}{4\pi} H_i U_k U^\alpha \bar{P}_{ik\alpha}$$

$$- \frac{1}{4\pi} U_n H_n H_{i\cdot jk} X_i X_j U_k - \frac{1}{4\pi} U_n H^\alpha H_{i,jk} x^i{}_\alpha X_j U_k = 0. \tag{3·23}$$

Further multiplying (3·18) by U_k we obtain

$$K_{33} + \rho\, U_k U_{r,rk} + U_k(\rho_{,k} U_{r,r} + U_{i,k}\, \rho_{,i} + 3\, \rho_{,k}) = 0. \tag{3·24}$$

Putting $l = \alpha$ and $m = 3$ in (3·20) we get

$$X_{\alpha 3} + \rho\, I_{\alpha 33} + J_{\alpha 3} - \frac{1}{4\pi} H\beta\, \bar{I}_{\alpha 3 \beta} - \frac{1}{4\pi} H_n H_{i,jk} x^i{}_\alpha X_j U_k = 0. \tag{3·25}$$

Furthermore multiplying (3·17) by $C_{i\alpha} C_{k3}$ we get

$$\Upsilon'_{\alpha 3} + \bar{I}_{\alpha 33} - H_\alpha U_k U_{r,rk} - H_n U_{i,jk} x^i{}_\alpha X_j U_k - H^\beta x^i{}_\alpha P_{ik\beta} U_k = 0, \tag{3·26}$$

where $\quad \Upsilon'_{\alpha 3} = C_{i\alpha} C_{k3} (U_{j,k} H_{i,j} - H_{j,k} U_{i,j} + H_{i,k} U_{r,r} + 2\, H_{i\cdot k}).$

Now we proceed to find the value of $U_{i,jk} x^i{}_\alpha X_j$, $U_{i,jk} X_i X_j$, $H_{i,jk} x^i{}_\alpha X_j$ and $H_{i,jk} X_i X_j$.

With the help of (2·10), (2·15), (3·9) and (3·12), the equation (3·17) gives

$$- U_n X_j H_{i,jk} + H_n X_j U_{i,jk} = \phi_{ik} + H_i U_{r,rk}, \tag{3·27}$$

where $\quad \phi_{ik} = U^\beta \bar{P}_{ik\beta} - H^\beta P_{ik\beta} + U_{j,k} H_{i,j} - H_{j,k} U_{i,j} + H_{i,k} U_{r,r} + 2\, H_{,i\cdot k}$.

Multiplging (3·27) by $x^i{}_\alpha$ and X_i we obtain respectively

$$- U_n H_{i,jk} x^i{}_\alpha X_j + H_n U_{i,jk} x^i{}_\alpha X_j = \phi_{ik} x^i{}_\alpha + H_\alpha U_{r,rk}, \tag{3·28}$$

$$- U_n H_{i,jk} X_i X_j + H_n U_{i,jk} X_i X_j = \phi_{ik} X_i + H_n U_{r,rk}. \tag{3·29}$$

Also, with the help of (2·10), (2·15), (3·9), (3·10) and (3·12), the equation (3·19) gives

$$- \rho\, U_n U_{i,jk} x^i{}_\alpha X_j + \frac{1}{4\pi} H_n H_{i,jk} x^i{}_\alpha X_j = \xi_{k\alpha}, \tag{3·30}$$

where

$$\xi_{k\alpha} = \rho\, x^i{}_\alpha U^\beta P_{ik\beta} - \frac{1}{4\pi} H^\beta x^i{}_\alpha \bar{P}_{ik\beta} - \frac{1}{4\pi} H_{j,k} H_{i,j} x^i{}_\alpha + \rho\, U_{j,k} U_{i,j} x^i{}_\alpha$$

$$+ \rho_{,k} U_j U_{i,j} x^i{}_\alpha + \rho_{,k} U_\alpha + Q_{k\alpha} + \rho U_{i,k} x^i{}_\alpha.$$

From (3·28) and (3·30) we obtain

$$H_{i,jk} x^i{}_\alpha X_j = U_{k\alpha} + U'_\alpha U_{r,rk}, \tag{3·31}$$

$$U_{i,jk} x^i{}_\alpha X_j = V_{k\alpha} + V'_\alpha U_{r,rk}, \tag{3·32}$$

where

$$U_{k\alpha} = \frac{4\pi\,(H_n \xi_{k\alpha} + \rho U_n x^i{}_\alpha \phi_{ik})}{H^2{}_n - 4\pi \rho U^2{}_n},$$

$$U'_\alpha = \frac{4\pi \rho U_n H_\alpha}{H^2{}_n - 4\pi \rho U^2{}_n},$$

$$V_{\lambda\alpha} = \frac{4\pi\ U_n\ \xi_{\lambda\alpha} + H_n\ \phi_{ik}\ x^i_\alpha}{H^2_n - 4\pi\ \rho U^2_n},$$

$$V'_\alpha = \frac{H_n\ H_\alpha}{H^2_n - 4\pi\ \rho U_n^2}.$$

Differentiating the equation (2·21), eliminating p_{ik} with the help of (3·19) and making use of (2·10), (2·15), (3·9), (3·12), (3·31) and (3·32) we obtain

$$U_{i,jk}\ X_i\ X_j = W_k + W'\ U_{r,rk}, \tag{3·33}$$

where

$$\begin{aligned} W_k = -\frac{1}{\rho U^2_n} \Big\{ & \rho_{,k}\ U^2_i + \rho U_i\ U_{i,k} + \rho_{,k}\ U_i\ U_j\ U_{i,j} \\ & + \rho U_i\ U_{i,j}\ U_{j,k} - \frac{1}{4\pi}\ H_{j,k}\ H_{i,j}\ U_i + \frac{1}{4\pi}\ H_{j,k}\ H_{j,i}\ U_i \\ & - p_{,i}\ U_{i,k} - \gamma p_{,k}\ (U_{r,r} + 3) + \rho U^\alpha\ U_i\ P_{ik\alpha} \\ & + \frac{1}{4\pi}\ (H_i\ U^\alpha - U_i\ H^\alpha)\ \bar{P}_{ik\alpha} + \rho U_n\ U^\alpha\ V_{k\alpha} \\ & + \frac{1}{4\pi}\ (U_n\ H^\alpha - H_n\ U^\alpha)\ U_{k\alpha} \Big\}, \end{aligned}$$

$$W' = -\frac{1}{\rho U^2_n}\Big\{ - \gamma\ p + \rho U_n\ U^\alpha\ V'_\alpha + \frac{1}{4\pi}(U_n\ H^\alpha - H_n\ U^\alpha)\ U'_\alpha \Big\}.$$

Now substituting from (3·33) in (3·29) we obtain

$$H_{i\ jk}\ X_i\ X_j = \frac{1}{U_n}\ (H_n\ W_k - \phi_{ik}\ X_i) + \frac{H_n}{U_n}\ (W' - 1)\ U_{r,rk}. \tag{3·34}$$

Thus the values of $U_{i,jk}\ x^i_\alpha\ X_j$, $U_{i,jk}\ X_i\ X_j$, $H_{i\ jk}\ x^i_\alpha\ X_j$ and $H_{i,jk}\ X_i\ X_j$ are known in terms of $U_{r,rk}$.

Now we are in a position to find the values of I_{i33}, J_{33}, K_{33} and $\bar{I}_{i33}$ in terms of $U_{r,rk}\ U_k$.

The relation (3·23) gives the value of J_{33} in terms of $U_k\ U_{r,rk}$. If we substitute this value of J_{33} in (3·20a) we get the value of I_{333} in terms of $U_k\ \ U_{r,rk}$. Also $\bar{I}_{333}$ and K_{33} are obtained in terms of $U_k\ U_{r,rk}$ from the relations (3·21) and (3·24) respectively. Further (3·25) gives the value of $I_{\alpha 33}$ and (3·26) gives the value of $\bar{I}_{\alpha 33}$ in terms of $U_k\ U_{r,rk}$.

To find the value of $U_k\ U_{r,rk}$ we put $l = m = r$ in (3·1), and obtain

$$U_k\ U_{r,rk} = I_{ia3}\ D_{ip}\ D_{\alpha p} + I_{i33}\ D_{ip}\ D_{3p}, \tag{3·35}$$

where we have used the relations

$$U_k\ D_{\alpha k} = 0 \text{ and } U_k\ D_{3k} = 1.$$

The first term in the right member of (3·35) is independent of $U_k\ U_{r,rk}$ and is completely known as is evident from (3·13), while the second term contains $U_k\ U_{r,rk}$ as the value of I_{i33} is known in terms of $U_k\ U_{r,rk}$. Thus (3·35) gives the value of $U_k\ U_{r,rk}$ in principle.

In this way the quantities I_{ijk}, J_{jk}, K_{jk} and $\bar{I}_{ijk}$ are completely known to us and consequently the second order partial derivatives of U_i, p^*, ρ and H_i are completely determined.

To find the expression for curvature and torsion of a streakline behind the shock surface, let s be the arc length of the streak-line measured from the shock surface and a_i be the coordinates of a point P just behind the shock surface on the streak-line, then we have[3]

$$\lambda_i = \frac{\partial a_i}{\partial s} = \frac{U_i}{q}\ ;\ q^2 = U_i\, U_i, \tag{3·36}$$

where λ_i are the components of the unit tangent vector to the streak-line at P.

Therefore

$$\frac{\partial^2 a_i}{\partial s^2} = \frac{\partial}{\partial a_j}\left(\frac{U_i}{q}\right)\lambda_j = \frac{U_{i,j}\lambda_j}{q} - U_l\, U_j\, U_{l,j}\,\frac{\lambda_i}{q^3}. \tag{3·37}$$

But from one of the Frenet's formulae we have

$$\frac{\partial^2 a_i}{\partial s^2} = \frac{\partial \lambda_i}{\partial s} = K\mu_i \tag{3·38}$$

where K is the curvature of the streak-line and μ_i are the components of the principal normal vector to the streak-line at P.

Thus the relation (3·37) becomes

$$K\mu_i = \frac{U_{i,j}U_j}{q^2} - \frac{U_l U_j\, U_{l,j}\,\lambda_i}{q^3}. \tag{3·39}$$

If we multiply both sides of this equation by μ_i we obtain

$$Kq^2 = U_{i,j}\, U_j\, \mu_i. \tag{3·40}$$

Eliminating μ_i between (3·39) and (3·40) we obtain

$$K^2 = \frac{(U_{i,j}\, U_j)^2}{q^4} - \frac{(U_i\, U_j\, U_{i,j})^2}{q^6}. \tag{3·41}$$

The torsion τ of the streak-line is given by[4]

$$\tau = -\frac{1}{K^2}\,\epsilon_{ijk}\,\frac{\partial a_i}{\partial s}\,\frac{\partial^2 a_j}{\partial s^2}\,\frac{\partial^3 a_k}{\partial s^3}. \tag{3·42}$$

where $\frac{\partial a_i}{\partial s}$ and $\frac{\partial^2 a_i}{\partial s^2}$ are given by (3·36) and (3·37) respectively

and

$$\begin{aligned}\frac{\partial^3 a_k}{\partial s^3} &= (U_m\, U_r\, U_{k,mr} + U_{k,m}\, U_{m,r}\, U_r)\,\frac{1}{q^3}\\ &- \frac{1}{q^5}\,(2\, U_m\, U_r\, U_{k,m}\, U_l\, U_{l,r} + U_m\, U_r\, U_{k,r}\, U_l\, U_{l,m}\\ &+ U_k\, U_m\, U_r\, U_l\, U_{l,mr} + U_k\, U_r\, U_{m,r}\, U_l\, U_{l,m}\\ &+ U_k\, U_r\, U_m\, U_{l,r}\, U_{l,m}) + \frac{4}{q^7}\, U_t\, U_{t,r}\, U_r\, U_k\, U_m\, U_l\, U_{l,m}.\end{aligned} \tag{3·43}$$

§ 4. DETERMINATION OF THE THIRD ORDER PARTIAL DERIVATIVES AND TORSION OF A VORTEX LINE BEHIND THE SHOCK SURFACE.

In this section we shall use a different matrix $\| E_{ij} \|$ to obtain the third order partial derivatives of U_i, though these results can also be obtaind by making use of the matrix $\| C_{ij} \|$ defined in section 2.

We get

$$E_{i\alpha} = x^i_{\alpha} \text{ and } E_{i3} = X_i. \tag{4·1}$$

If det $E_{ij} \neq 0$, then we can define F_{ij} by the relations

$$F_{ij} = \frac{\text{Cofactor of } E_{ji} \text{ in } \|E_{pq}\|}{|E_{pq}|}, \quad F_{ik} E_{kj} = \delta_{ij}\,;\; F_{ki} E_{jk} = \delta_{ij}. \tag{4·2}$$

Further we define the quantities S_{ijkr} by

$$S_{ijkr} = U_{l,mnp} E_{li} E_{mj} E_{nk} E_{pr}, \tag{4·3}$$

which, in consequence of (4·2), gives

$$U_{l,mnp} = S_{ijkr} F_{il} F_{jm} F_{kn} F_{rp}. \tag{4·4}$$

Furthermore, if we differentiate (3·1) to (3·4) with respect to y^{α} we obtain

$$H_{l,mnr} x^r_{\alpha} = \psi_{lmn\alpha}, \tag{4·5}$$

$$U_{l,mnr} x^r_{\alpha} = \psi'_{lmn\alpha}, \tag{4·6}$$

$$p^*_{l,mnr} x^r_{\alpha} = \psi''_{mn\alpha}, \tag{4·7}$$

$$\rho_{,mnr} x^r_{\alpha} = \psi'''_{mn\alpha}, \tag{4·8}$$

where

$$\psi_{lmn\alpha} = t \frac{\partial}{\partial y^{\alpha}} (T_{ijk} D_{il} D_{jm} D_{kn}),$$

$$\psi'_{lmn\alpha} = t \frac{\partial}{\partial y^{\alpha}} (I_{ijk} D_{il} D_{jm} D_{kn}),$$

$$\psi''_{mn\alpha} = t \frac{\partial}{\partial y^{\alpha}} (J_{jk} D_{jm} D_{kn}),$$

$$\psi'''_{mn\alpha} = t \frac{\partial}{\partial y^{\alpha}} (K_{jk} D_{jm} D_{kn}).$$

Thus $\psi_{lmn\alpha}$, $\psi'_{lmn\alpha}$, $\psi''_{mn\alpha}$ and $\psi'''_{mn\alpha}$ are known to us in principle.

Now, by virtue of (4·6), the relation (4·3) gives

$$S_{ijk\alpha} = \psi'_{lmn\alpha} E_{li} E_{mj} E_{nk}, \tag{4·9}$$

which gives the value of $S_{ijk\alpha}$ in principle. Because of the symmetry conditions satisfied by the quantities S_{ijkr}, we have now to determine $S_{\alpha 333}$ and S_{3333} only. From (4·3) we obtain

$$S_{\alpha 333} = U_{l,mnp} x^l_{,\alpha} X_m X_n X_p, \tag{4·10}$$

where we have used the relations (4·1).

Furthermore the relation (4·3) gives

$$S_{3333} = U_{l,mnp} X_l X_m X_n X_p. \tag{4·11}$$

Now we proceed to find the values of $U_{l,mnp}\, x^l_{\alpha} X_m$ and $U_{l,mnp} X_l X_m$.

Differentiating (3·17) with respect to a_m and using (2·10), (2·15), (4·5) and (4·6) we obtain

$$H_n \, U_{i,jkm} \, X_j - U_n \, H_{i,jkm} \, X_j = \phi_{ikm} + H_i \, U_{r,rkm}, \tag{4·12}$$

where

$$\begin{aligned}\phi_{ikm} = {} & U^\beta \, \psi_{ikm\beta} - H^\beta \, \psi'_{ikm\beta} + 2 \, H_{i,km} \\ & + U_{j,km} \, H_{i,j} + U_{j,k} \, H_{i,jm} + U_{j,m} \, H_{i,jk} \\ & - H_{j,km} \, U_{i,j} - H_{i,k} \, U_{i,jm} - H_{j,m} \, U_{i,jk} \\ & + H_{i,km} \, U_{r,r} + H_{i,k} \, U_{r,rm} + H_{i,m} \, U_{r,rk}.\end{aligned}$$

Multiplying (4·12) by x^i_α and X_i we obtain respectively

$$H_n \, U_{i,jkm} \, x^i_\alpha \, X_j - U_n \, H_{i\,jkm} \, x^i_\alpha \, X_j = x^i_\alpha \, \phi_{ikm} + H_\alpha \, U_{r,rkm}. \tag{4·13}$$

and

$$H_n \, U_{i,jkm} \, X_i \, X_j - U_n \, H_{i,jkm} \, X_i \, X_j = X_i \, \phi_{ikm} + H_n \, U_{r,rkm} \tag{4·14}$$

Further, differentiating (3·19) with respect to a_m, multiplying the resultant equation by x^i_α and using (2·10), (2·15), (4·5), (4·6), (4·7) we obtain

$$\begin{aligned} & - \rho \, U_n \, U_{i,jkm} \, x^i_\alpha \, X_j \\ & \quad + \frac{1}{4\pi} H_n \, H_{i,jkm} \, x^i_\alpha \, X_j = \xi_{km\alpha}, \end{aligned} \tag{4·15}$$

where

$$\begin{aligned}\xi_{km\alpha} = {} & \rho U^\beta \, \psi'_{ikm\beta} \, x^i_\alpha - \frac{1}{4\pi} H^\beta \, \psi_{ikm\beta} \, x^i_\alpha \quad \psi''_{km\alpha} \\ & + x^i_\alpha \, \{ (\rho U_i)_{,km} + (\rho U_j)_{,km} \, U_{i,j} + (\rho U_j)_{,k} \, U_{i,jm} \\ & + (\rho U_j)_{,m} \, U_{i,jk} - \frac{1}{4\pi} \, H_{j,km} \, H_{i,j} - \frac{1}{4\pi} H_{j,k} \, H_{i,jm} - \frac{1}{4\pi} H_{j,m} \, H_{i,jk} \}.\end{aligned}$$

From (4·13) and (4·15) we obtain

$$H_{i,jkm} \, x^i_\alpha \, X_j = R_{km\alpha} + R'_\alpha \, U_{r,rkm}, \tag{4·16}$$

and

$$U_{i,jkm} \, x^i_\alpha \, X_j = Q_{km\alpha} + Q'_\alpha \, U_{r,rkm}, \tag{4·17}$$

where

$$R_{km\alpha} = \frac{4\pi \, (H_n \, \xi_{km\alpha} + \rho U_n \, x^i_\alpha \, \phi_{ikm})}{H^2_n - 4\pi \, \rho U^2_n},$$

$$R'_\alpha = \frac{4\pi \, \rho U_n \, H_\alpha}{H^2_n - 4\pi \, \rho U^2_n},$$

$$Q_{km\alpha} = \frac{4\pi \, U_n \, \xi_{km\alpha} + H_n \, x^i_\alpha \, \phi_{ikm}}{H^2_n - 4\pi \, \rho U^2_n},$$

$$Q'_\alpha = \frac{H_n \, H_\alpha}{H^2_n - 4\pi\rho \, U^2_n}.$$

Furthermore, differentiating the equation (3·22) we get

$$p^*_{,ikm}\, U_i + p^*_{,ik}\, U_{i,m} + (p^*_{,i}\, U_{i,k})_{,m} - \frac{1}{4\pi}\, \{(H_j\, U_i)_{,k}\, H_{j,i}\}_{,m} - \frac{1}{4\pi}\, (H_j\, U_i)_{,m}\, H_{j,ik}$$

$$+ \gamma\, \{ p_{,k}\, (U_{r,r}+3) \}_{,m} + \gamma\, p_{,m}\, U_{r,rk} - \frac{1}{4\pi}\, H_j\, U_i\, H_{j,ikm} + \gamma p\, U_{r,rkm} = 0. \qquad (4\cdot18)$$

Also, differentiating (3·19) with respect to a_m, multiplying throughout by U_i and eliminating $p^*_{,ikm}\, U_i$ between the resultant equation and (4·18) we obtain

$$U_i\, \{ (\rho U_i)_{,km} + (\rho U_j)_{,km}\, U_{i,j} + (\rho U_j)_{,k}\, U_{i,jm}$$

$$+ (\rho U_j)_{,m}\, U_{i,jk} - \frac{1}{4\pi}\, H_{j,km}\, H_{i,j} - \frac{1}{4\pi}\, H_{j,k}\, H_{i,jm} - \frac{1}{4\pi}\, H_{j,m}\, H_{i,jk} \} - p^*_{,ik}\, U_{i,m}$$

$$- (p^*_{,i}\, U_{i,k})_{,m} + \frac{1}{4\pi}\, \{(H_j\, U_i)_{,k}\, H_{j,i} \}_{,m} + \frac{1}{4\pi}\, (H_j\, U_i)_{,m}\, H_{j,ik}$$

$$- \gamma\, \{p_{,k}\, (U_{r,r} + 3)\}_{,m} - \gamma p_{,m}\, U_{r,rk} + \rho U_i\, U_j\, U_{i,jkm} - \frac{1}{4\pi}\, U_i\, H_j\, H_{i,jkm}$$

$$+ \frac{1}{4\pi}\, U_i\, H_j\, H_{j,ikm} - \gamma\, p\, U_{r,rkm} = 0. \qquad (4\cdot19)$$

By virtue of (2·10), (2·15), (4·5) and (4·6) the equation (4·19) assumes the form

$$\rho U^2_n\, U_{i,jkm}\, X_i\, X_j + \rho U_n\, U^\alpha\, U_{i,jkm}\, x^i_{\ \alpha}\, X_j + \frac{1}{4\pi}\, (U_n\, H^\alpha - H_n\, U^\alpha)\, H_{i,jkm}\, x^i_{\ \alpha}\, X_j$$

$$- \gamma p\, U_{r,rkm} + T_{km} = 0, \qquad (4\cdot20)$$

where
$$T_{km} = U_i\, \{ (\rho U_i)_{,km} + (\rho U_j)_{,km}\, U_{i,j} + (\rho U_j)_{,k}\, U_{i,jm}$$

$$+ (\rho U_j)_{,m}\, U_{i,jk} - \frac{1}{4\pi}\, H_{j,km}\, H_{i,j} - \frac{1}{4\pi}\, H_{j,k}\, H_{i,jm} - \frac{1}{4\pi}\, H_{j,m}\, H_{i,jk} \} - p^*_{,ik}\, U_{i,m}$$

$$- (p^*_{,i}\, U_{i,k})_{,m} + \frac{1}{4\pi}\, \{(H_j\, U_i)_{,k}\, H_{j,i}\}_{,m} + \frac{1}{4\pi}(H_j\, U_i)_{,m}\, H_{j,ik} - \gamma\, \{ p_{,k}\, (U_{r,r} + 3) \}_{,m}$$

$$- \gamma p_{,m}\, U_{r,rk} + \rho U_i\, U^\alpha\, \psi'_{ikm\alpha} - \frac{1}{4\pi}\, U_i\, H^\alpha\, \psi_{ikm\alpha} + \frac{1}{4\pi}\, H_j\, U^\alpha\, \psi_{jkm\alpha}.$$

With the help of (4·16) and (4·17), the equation (4·20) gives

$$U_{i,jkm}\, X_i\, X_j = - \frac{1}{\rho U^2_n}\, \{ \rho U_n\, U^\alpha\, Q_{km\alpha} + T_{km} + \frac{1}{4\pi}\, (U_n\, H^\alpha - H_n\, U^\alpha)\, R_{km\alpha} \}$$

$$+ \frac{1}{\rho U^2_n}\, \{ \gamma p + \frac{1}{4\pi}\, (H_n\, U^\alpha - U_n\, H^\alpha)\, R'_\alpha - \rho U_n\, U^\alpha\, Q'_\alpha\}\, U_{r,rkm} \qquad (4\cdot21)$$

Now by virtue of (4·2), (4·10), (4·11), (4·17) and (4·21), the relation (4·4) gives the value of $U_{r,rkm}\, X_k\, X_m$:

$$U_{r,rkm}\, X_k\, X_m\, (1 - F_{\alpha p}\, F_{3p}\, R'_\alpha - F_{3p}\, F_{3p}\, \overline{R}') = S_{i\alpha 33}\, F_{ip}\, F_{\alpha p}$$
$$+ F_{\alpha p}\, F_{3p}\, R_{ls\alpha}\, X_l\, X_s + F_{3p}\, F_{3p}\, \overline{P}_{ls}\, X_l\, X_s, \qquad (4\cdot22)$$

where

$$\bar{R}_{ls} = -\frac{1}{\rho U^2_n} \{ \rho U_n U^\alpha Q_{ls\alpha} + T_{ls} + \frac{1}{4\pi} (U_n H^\alpha - H_n U^\alpha) R_{ls\alpha} \}$$

and

$$\bar{R}' = \frac{1}{\rho U^2_n} \{\gamma p + \frac{1}{4\pi} (H_n U^\alpha - U_n H^\alpha) R'_\alpha - \rho U_n U^\alpha Q'_\alpha \}.$$

Thus if we substitute for $U_{l;mnp}\, x^l_\alpha\, X_m$ from (4·17) in (4·10) and make use of (4·22) we get the value of $S_{\alpha 333}$. Similarly, with the help of (4·21) and (4·22), the relation (4·11) gives the value of S_{3333}. In this way the quantities S_{ijkr} have been completely determined. Therefore the third order partial derivatives of U_i are known from (4·4).

Now, let the vorticity vector at a point $P(x_i)$ on the vortex line just behind the shock surface be given by w_i. Then

$$w_i = \epsilon_{ijk} \frac{\partial U_k}{\partial x_j} = \frac{1}{t} \epsilon_{irs} U_{s,r}. \tag{4·23}$$

Differentiating (4·23) we obtain

$$w_{i,j} = \frac{1}{t} \epsilon_{irs} U_{s,rj}, \tag{4·24}$$

and further differentiation gives

$$w_{i,jk} = \frac{1}{t} \epsilon_{irs} U_{s,rjk}. \tag{4·25}$$

Since we have already calculated the values of $U_{s,r}$, $U_{s,rj}$ and $U_{s,rjk}$, therefore, the above relations give the values of w_i, $w_{i,j}$ and $w_{i,jk}$ in terms of known quantities in principle.

Now we proceed to find the curvature and torsion of the vortex line at $P(x_i)$.

Let $\bar{s}$ be the distance of P from the shock surface measured along the vortex line under consideration. Then we have from the definition of the vortex line

$$\bar{\lambda}_i = \frac{\partial x_i}{\partial \bar{s}} = \frac{w_i}{w} \; ; \; w^2 = w_i w_i, \tag{4·26}$$

where $\bar{\lambda}_i$ are the component of the unit tangent vector to the vortex line at $P\ (x_i)$. Therefore,

$$\begin{aligned} \frac{\partial^2 x_i}{\partial \bar{s}^2} &= \frac{1}{t} \frac{\partial}{\partial a_j} \left(\frac{w_i}{w} \right) \bar{\lambda}_j \\ &= \frac{1}{t} \left(\frac{w_{i,j} \bar{\lambda}_j}{w} - \frac{w_l w_j w_{l,j} \bar{\lambda}_i}{w^3} \right). \end{aligned} \tag{4·27}$$

But from one of the Frenet's formulae we have

$$\frac{\partial^2 x_i}{\partial \bar{s}^2} = \frac{\partial \bar{\lambda}_i}{\partial \bar{s}} = \bar{K} \; \bar{\mu}_i .$$

where $\bar{K}$ is the curvature of the vortex line and $\bar{\mu}_i$ are the components of the principal normal vector at $P(x_i)$.

Therefore the relation (4·27) assumes the form

$$\bar{K} \; \mu_i = \frac{1}{t}\left(\frac{w_{i,j} \; w_j}{w^2} - \frac{w_l \; w_j \; w_{l,j} \; \bar{\lambda}_i}{w^3} \right) \tag{4·28}$$

If we multiply both sides of (4·28) by $\bar{\mu}_i$ we obtain

$$\bar{K} = \frac{1}{t} \; \frac{w_{i,j} \; w_j \; \bar{\mu}_i}{w^2} . \tag{4·29}$$

Eliminating $\bar{\mu}_i$ between (4·28) and (4·29) we obtain

$$\bar{K}^2 = \frac{1}{t^2 \; w^2} \left\{ \frac{(w_{i,j} \; w_j)^2}{w^2} - \frac{(w_i \; w_j \; w_{i,j})^2}{w^4} \right\} . \tag{4·30}$$

The torsion $\bar{\tau}$ of the vortex line at $P(x_i)$ is given by[4]

$$\bar{\tau} = - \frac{1}{\bar{K}^2} \; \varepsilon_{ijk} \; \frac{\partial x_i}{\partial \bar{s}} \; \frac{\partial^2 x_j}{\partial \bar{s}^2} \; \frac{\partial^3 x_k}{\partial \bar{s}^3} , \tag{4·31}$$

where $\frac{\partial x_i}{\partial \bar{s}}$ and $\frac{\partial^2 x_j}{\partial \bar{s}^2}$ are respectively given by the relations (4·26) and (4·27). The expression for $\frac{\partial^3 x_k}{\partial \bar{s}^3}$ can be easily obtained by inspection of (4·27), (3·37) and (3·43).

REFERENCES

1. Fletcher, A. H., Taub, A. H. and Bleakney, W. The Mach reflection of shock waves at nearly glancing incident. *Rev Modern Physics* 23 : 271–286, (1951).
2. Mishra, R. S. Determination of jump conditions for three dismensional shocks in conducting fluids, (unpublished).
3. Taub, A. H. Determination of flows behind stationary and pseudo-stationary shocks, *Ann. of Math.*, 62 : 300–325, (1955).
4. Eisenhart, L. P. An introduction to differential geometry with use of the tensor calculus, *Princeton University Press*, (1940).

A NEW CONCEPT OF THE STRUCTURE OF POLYTROPIC CONFIGURATIONS

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ABSTRACT

It has generally been believed that in a complete polytrope, the pressure-density relation : $P = K\rho^{1+1/n}$ is relevant for a definite value of n, at every point in the configuration. In this paper author has shown that at the origin, the pressure-density relation, whatever be the index of the configuration, is valid only either for $n = 0$ or for $n = -1$. The value of n for which the pressure-density relation is valid in the rest of the parts of the configuration is the same as that of the index of the configuration. The immediate neighbourhood of the origin is an interfacial region, in which two sets of equations : one governing the origin and the other governing the rest of the parts of the configuration, are valid simultaneously.

INTRODUCTION

Polytropic configurations are the gaseous configurations in equilibrium under their own gravitation. Equations governing such cofigurations are (Chandrashekhar, S., 1939, eq. 6 & 7, p. 87).

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho\,;\ \frac{dM(r)}{dr} = 4\pi r^2\rho, \tag{1}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho. \tag{2}$$

Substitutions

$$\rho = \lambda\theta^n\,;\ P = K\rho^{1+1/n} = K\lambda^{1+1/n}\,\theta^{n+1}, \tag{3}$$

transform equation (2) into

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \tag{4}$$

where ξ is related with r by

$$r = \alpha\xi\,;\ \alpha = \left[\frac{(n+1)K}{4\pi G}\,\lambda^{1/n-1}\right]^{\frac{1}{2}}. \tag{5}$$

Solutions of (4) can be classified into three kinds (Russell, H. N., 1931) :

(*a*) E–solutions, with θ finite and a maximum at the centre.

(*b*) M–solutions, with θ infinite and a maximum at the centre.

(*c*) E–solutions, in which θ is zero for some finite value of ξ increases to a maximum and then diminishes.

Solutions of (4), finite at the origin, satisfy the boundary conditions

$$\theta = 1\,;\ \frac{d\theta}{d\xi} = 0 \text{ at } \xi = 0. \tag{6}$$

PHYSICAL CONDITIONS AT THE ORIGIN AND IN THE IMMELIATE NEIGHBOURHOOD OF THE ORIGIN

Equations governing the configurations suggest that it is necessary to assume a relation between pressure and density for the study of the structure of the configurations. Of course, we have to assume the relation such that it must be consistent with equations in (1). Assume an arbitrary small volume at a distance r from the centre ; then first equation in (1) can be written as

$$\frac{dP}{dr} = -\tfrac{4}{3}\pi r \,\bar{\rho}\, G, \tag{7}$$

where $\bar{\rho}$ is the value of ρ at the point considered. Equation (7) shows that dP/dr vanishes at the origin, for $\bar{\rho}$ at the origin is a finite quantity.

Let our pressure-density relation be $P = K\rho^{1+1/n}$, where n is a constant. To decide the particular value of n for which the assumed pressure-density relation will be consistent with equations in (1), we proceed as follows : Assume an arbitrary region in which dP/dr vanishes at every point. Obviously then, what is true at every point in such a region is also true at the origin – situation at every point being the same. In such regions, clearly, $P = K\rho^{1+1/n}$ can be valid only for $n = -1$ (Author is grateful to Professor S. Chandrashekhar of Chicago for his correspondance in which he incidentally pointed out that Emden's equation for $n = -1$ is valid only in those parts of the configurations in which pressure is constant without density vanishing simultaneously).

If we consider an arbitrary small volume in $(r\,;\,\rho)$-plane, then the first equation in (1) can be expressed as

$$\frac{d\rho}{dr} = -\frac{4\pi n G r}{3(n+1)}\rho^{-1/n}. \tag{8}$$

Abovegoing clears that whatever be the value of n ($n = 0$ and $n = -1$ are to be excluded.) since ρ is finite at the origin, $d\rho/dr$ vanishes at the origin. Now assume a small region in which $d\rho/dr$ vanishes at every point. The pressure-density relation can be intepreted in the form

$$\rho = \left[\frac{P}{K}\right]^{\frac{n}{n+1}}. \tag{9}$$

Equation (9) suggests that in a region where $d\rho/dr$ vanishes at every point, *i.e.* ρ is a constant at every point, the pressure-density relation can be relevant only for $n = 0$.

We may, therefore, conclude that if we consider equations in $(r\,;\,P)$ – plane then solutions for $n = -1$ will govern the physical conditions at the origin, whatever be the index of the configuration. If we consider the solutions in $(r\,;\,\rho)$ – plane, then solutions for $n = 0$ will govern the origin.

For $n = 0$ and $n = -1$, pressure density relation gives $\rho = 1$ and $P = K$ respectively. These equations cannot be taken as functional relations between P and ρ and, therefore, cannot be taken as consistent with equations in (1). Actually $n = 0$ and $n = -1$ are to be regarded as the limiting cases of the pressure-density relation. We have to find mathematically relevant solutions for n tending to zero and to minus one. For n tending to 0 and -1, P and ρ remain related *i.e.* neither P nor ρ is a constant ; hence no inconsistency with equations of hydrostatic equilibrium arises.

As we leave the origin, we see that equations governing the configuration do not impose any further condition. In the rest of the parts of the configurations, therefore, we can take pressure-density relation relevant for different values of n for the study of different configurations. Of course, the conditions at the origin in all the configurations remain the same.

We have seen that in a polytropic configuration, of a particular index, the origin and the rest of the parts of the configurations are governed by solutions of Emden's equation for two different values of n. As there is no discontinuity at the origin in a configuration, the immediate neighbourhood of the origin is an interfacial region in which two sets of equations : one governing the origin and the other governing the rest of the parts of the configuration are valid simultaneously.

SOLUTIONS FOR $n=0$ AND $n=-1$

It has recently been shown (Shambhunath Srivastava, 1966) that equation (4) is irrelevant for $n = 0$ and $n = -1$. Relevant forms of the solutions of equations, equivalent to equation (4), can be considered as follows : We have seen that $n = 0$ and $n = -1$, have to be regarded as limiting cases of the relation $P = K\rho^{1+1/n}$. Therefore, if we eliminate one of the dependent variables between $P = K\rho^{1+1/n}$ and equation (2), then the solutions of the resulting equation can be significant only if the process of elimination involves some mathematical operation over the pressure-density relation. Of course, the resulting equation, even if the solutions be insignificant for these two values of n *i.e.* even if the process of eliminating one dependent variable from (2) does not involve any mathematical operation over $P = K\rho^{1+1/n}$, can be taken relevant for further applications.

Eliminating one dependent variable between equation (2) and $P=K\rho^{1+1/n}$, we get

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\,\rho^{1/n-1}\,\frac{d\rho}{dr}\right) = -\frac{4\pi nG}{n+1}\,\rho, \tag{10}$$

and

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\,P^{-n/(n+1)}\,\frac{dP}{dr}\right) = -4\pi GK^{-2n/(n+1)}\,P^{n/(n+1)} \tag{11}$$

Elimination of ρ does not involve any mathematical operation over $P = K\rho^{1+1/n}$. Hence as explained above, solutions of equation (11) cannot be significant. Elimination of P involves a mathematical operation over $P = K\rho^{1+1/n}$, but the form of the resulting equation remains indeterminate for $n = 0$ and $n = -1$. Of course, both equations (10) and (11) remain relevant for further applications.

It can be shown in the usual manner (Chandrashekhar S., 1939 p. 105)

$$U = -\frac{4\pi nG}{n+1}\,\frac{r\rho^{2-1/n}}{\rho'}\;;\; V = -\frac{r\rho'}{\rho}, \tag{12}$$

and

$$u = -4\pi G\,\frac{rP^{2n/(n+1)}}{P'}\;;\; v = -\frac{rP'}{P}\,. \tag{13}$$

reduce equations (10) and (11) to the first order equations

$$\frac{UdV}{VdU} = -\frac{nU + V - n}{n(U + V - 3)}, \tag{14}$$

and

$$\frac{udv}{vdu} = -\frac{(n+1) - v - (n+1)u}{3(n+1) - nv - (n+1)u}, \tag{15}$$

respectively. Equation (14) becomes indeterminate for $n = 0$ and for $n = -1$, the integration of (14) is complicated. Equation (15) gives finite solutions for $n = 0$ and $n = -1$.

For $n = 0$, equation (15) becomes

$$\frac{dv}{du} + \frac{v(1-u)}{(3-u)u} = \frac{v^2}{(3-u)u}. \tag{16}$$

Foregoing is reducible to a linear form and the primitive is

$$\frac{1}{v}\,\frac{1}{(u-3)^{2/3}\,u^{1/3}} = \int \frac{du}{(u-3)^{5/3}\,u^{4/3}} + c. \tag{17}$$

With the substitution

$$u = 1/t,$$

we get

$$\int \frac{du}{(u-3)^{5/3}\,u^{4/3}} = -\int \frac{t\,dt}{(1-3t)^{5/3}}. \tag{19}$$

With further substitutions

$$1 - 3t = A \quad ; \quad -3dt = dA, \tag{20}$$

the right-hand side of (19) becomes integrable and we get

$$\int \frac{du}{(u-3)^{5/3}\,u^{4/3}} = -\left(\frac{A^{-2/3}}{6} + \frac{A^{1/3}}{3}\right). \tag{21}$$

Substituting (21) into (19) and reverting to the original variables, we get the complete primitive

$$\frac{1}{v}\,\frac{1}{(u-3)^{2/3}\,u^{1/3}} = -\tfrac{1}{6}\,\frac{u^{2/3}}{(u-3)^{2/3}} - \tfrac{1}{3}\,\frac{(u-3)^{1/3}}{u^{1/3}} + c. \tag{22}$$

With a simple re-arrangement, the abovegoing can be re-expressed as

$$v = \frac{2}{(2-u) + 2c\,(u-3)^{2/3}\,u^{1/3}}, \tag{23}$$

where c is a constant of integration. When $c = 0$, the curve is a hyperbola.

The variables introduced in (12) and (13) have a great advantage that positive quadrants ($u \geqslant 0$; $v \geqslant 0$) contain only such parts of (r ; P) - and (r ; ρ) - solutions, respectively, which are of astrophysical interest (Chandrashekhar S. 1939, p. 105).

When $c = 0$, the part of the curve which has an astrophysical validity starts from $v = 1$ and approaches the line $u = 2$ asymptotically.

For $n = -1$, equation (15) becomes

$$\frac{u}{v}\frac{dv}{du} = 1, \tag{24}$$

Equation (24) integrates to (25)

$$v = c'u,$$

where c' is a constant of integration. Equation (25) is a straight line inclined at an angle $\tan^{-1} c'$ to the u axis. Solution is valid physically when $c' \geqslant 0$. Equations in (12) and (13) suggest that u and U are related and v and V are related. If we substitute P and P' in (13) in terms of r and ρ and eliminate r and ρ between the resulting equations ; then we get

$$u = K^{(n-1)/(n+1)}U \, ; \; v = K^{(n+1)/n}V \tag{26}$$

For $n = 0$, $u = K^{-1}U$ but v cannot be expressed as $f(V)$. For $n = -1$, $u \neq f(U)$ and $v = 0$. Hence clearly, solutions in $(U : V)$- plane corresponding to that of (23) and (25) do not exist. Thus (23) and (25) are the only solutions which govern the origin of a complete polytrope.

We have seen that in a complete polytrope the origin of the configuration is governed only by solutions for $n = 0$ and $n = -1$; and we can study these solutions only if we consider solutions in $(r \,;\, P)$- plane. Thus we see that r and P are the only suitable variables which can give a clear concept of the structure of the polytropic configurations.

INTERFACIAL REGION

We have seen that in a complete polytrope, the origin and the rest of the parts of the configurations are governed by solutions of equations equivalent to the Lane-Emden equation of index n, for two different values of n. Whatever be the index of the configuration, if we consider structure in $(r \,;\, P)$- plane the origin is governed only by solutions for $n = -1$, As there is no discontinuity at the origin, there subsists an interfacial region in which two sets of equations : one governing the origin and the other governing the rest of the parts of configurations are valid simultaneously. It is easy to see that mass and the radius of the configuration, in $(r \,;\, P)$- variables are given by

$$M = \left[\frac{K^{2n/(n+1)}}{G\sqrt{4\pi G}}\, P^{-n/(n+1)}\, r^2 \frac{dP}{dr}\right]_{r=r_1} \tag{27}$$

and

$$R = \left[\frac{K^{2n/(n+1)}}{4\pi G}\right]^{\frac{1}{2}} r_1 \tag{28}$$

respectively, where solutions have their first zero at $r = r_1$. Potential energy is given by

$$\Omega = \int_0^R 12\pi r^2 P dr. \tag{29}$$

Let $P = K\rho^{1+1/n}$ be relevant for n_0 and n at the origin and in the rest of the parts of the configurations. At the interface the values of ρ, M, R, and Ω for the two mentioned sets of variables should be identical. Let $(r_0 ; P_0)$ – and $(r ; P)$ - be the variables used for the two respective regions. Then the 'equations of fit' are

$$\left[\frac{P_0}{K}\right]^{n_0/(n_0+1)} = \left[\frac{P}{K}\right]^{n/(n+1)} \tag{30}$$

$$K^{2n_0/(n_0+1)}\, P_0^{-n_0/(n_0+1)}\, r_0^2 \frac{dP_0}{dr_0} = K^{2n/(n+1)} \cdot P^{-n/(n+1)}\, r^2 \frac{dP}{dr}, \tag{31}$$

$$K^{n_0/(n_0+1)} r_0 = K^{n/(n+1)}\, r, \tag{32}$$

$$P_0 r_0^2 = Pr^2. \tag{33}$$

Raise equation (32) to third power, multiply by (30) and divide by (31), we are left with

$$\frac{r_0 P_0^{2n_0/(n_0+1)}}{P'_0} = \frac{rP^{2n/(n+1)}}{P'}. \tag{34}$$

The abovegoing can be re-expressed as

$$u\,(n_0\,;\,r_0) \quad \equiv \quad u\,(n\,;\,r). \tag{35}$$

Further multiply (31) by (30) and (32) and divide by (33), we get

$$K^{2n_0/(n_0+1)} \frac{r_0 P'_0}{P_0} = K^{2n/(n+1)} \frac{rP'}{P}. \tag{36}$$

Equation (36) can be re-written as

$$Kv(n_0\,;\,r_0) = V(n_0\,;\,r_0) \quad \equiv \quad V(n\,;\,r) = Kv(n\,;\,r). \tag{37}$$

Thus our equation of fit are

$$u\,(n_0\,;\,r_0) = u\,(n\,;\,r) \quad ; \quad V\,(n_0\,;\,r_0) = V\,(n\,;\,r). \tag{38}$$

We already know the solutions in $(u\,;\,v)$ plane. Solutions in our present $(u\,;\,V)$ – plane can be obtained very simply by multiplying ordinate in $(u\,;\,v)$ - plane by $K^{2n/(n+1)}$. Thus we see that all characteristic features in $(u\,;\,v)$ – plane are retained in $(u\,;\,V)$ – plane.

A homologous family in $(r : P)$ – plane yields only one curve in $(u\,;\,v)$ – plane. Let $u\,(n\,;\,r)\,;\,v\,(n\,;\,r) \equiv E\,(n)$ – curve. A point of intersection of $E(n)$ – curve and equation (25) corresponds to a particular solution of equations of fit. Substituting P and P' as $f(r)$ and the numerical values of u and v obtained by solutions of, $E\,(n)$ – curve and equation (25), we get an equation in ξ the roots of which correspond to the interface.

As soon as we leave the interface, solutions of Emden's equation for different values of n become relevant in different parts of configurations. Mass and radius of the configurations can be obtained from equations (27) and (28). The value of K is given by (Chandrashekhar S, 1939, p. 86)

$$K = \frac{k}{\mu H}\,\theta_{\gamma'} \tag{39}$$

where notations in (39) have the same meaning as used by Chandrashekhar.

CONCLUSIONS

1. If we consider equations in $(r\,;P)$ - plane, equivalent to the Emden's equation, then solutions for $n=-1$ will govern the physical conditions at the origin, whatever be the index of the polytrope. If we consider solutions in $(r\,;\rho)$ - plane, than solutions for $n=0$ will govern the origin.

2. Emden's equation in $(r\,;P)$-plane, when transformed in $(u\,;v)$-variables, give finite solutions for $n=0$ and $n=-1$. These solutions are given in (23) and (25). Solutions corresponding to these solutions do not exist in $(U\,;V)$ - plane.

3. r and P are the most suitable variables for the study of the structure of the polytropic configurations. Solutions (23) and (25) govern the origin. Equation (38) governs the interfacial region. The rest of the parts of the configurations are governed by solutions of (11) for some different definite values of n.

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REFERENCES

Chandrashekhar, S. An Introduction to the Study of Stellar Structure. *Chicago : University of Chicago Press* : *Chap.* iv, (1939).

Russell, H. N. M. N., **91** : 741, (1931).

Srivastava, Shambhunath. *National Academy of Sciences, Ind.*, 36-A : 883-884, (1966).

ON EXTENSION OF BATEMAN'S FUNCTION AND AN ALLIED FUNCTION

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ABSTRACT

In this paper a few properties of Bateman's function $K_n(x)$ are given and extensions of this function and an allied function are studied.

1. INTRODUCTION

Bateman defined[1] the function $K_n(x)$ by

$$K_n(x) = \frac{2}{\pi} \int_0^{\pi/2} \cos (x \tan \theta - n\theta) \, d\theta, \tag{1}$$

the fundamental properties of this function are

$$(n-2)\, K_{n-2}(x) - (n+2)\, K_{n+2}(x) = 4x\, K_n'(x), \tag{2}$$

$$(n-2)\, K_{n-2}(x) + (n+2)\, K_{n+2}(x) + 2(n-2x)\, K_n(x) = 0, \tag{3}$$

$$K_{n-1}'(x) + K_{n+1}'(x) = K_{n-1}(x) - K_{n+1}(x), \tag{4}$$

and

$$x\, K_n''(x) + (n-x)\, K_n(x) = 0. \tag{5}$$

When n is a positive integer or zero then

$$K_{2n}(x) = \frac{(-1)^n}{n\,!}\, x e^x \frac{d^n}{dx^n} (e^{-2x} x^{n-1}) \tag{6}$$

Later on, Srivastava[3] defined an allied function $T_n(x)$ given by

$$T_n(x) = \frac{2}{\pi} \int_0^{\pi/2} \sin (x \tan \theta - n\, \theta) \, d\theta. \tag{7}$$

He studied some of the properties of this function independently and in association with the function $K_n(x)$. The fundamental properties for this function are

$$(n-2)\, T_{n-2}(x) - (n+2)\, T_{n+2}(x) = 4x\, T_n'(x), \tag{8}$$

$$(n-2)\, T_{n-2}(x) + (n+2)\, T_{n+2}(x) + 2(n-2x)\, T_n(x) = -\frac{8}{\pi}, \tag{9}$$

$$T_{n-1}'(x) + T_{n+1}'(x) = T_{n-1}(x) - T_{n+1}(x), \tag{10}$$

and

$$x\, T_n''(x) + (n-x)\, T_n(x) + \frac{2}{\pi} = 0. \tag{11}$$

2. Results connected with $K_n(x)$

Multiplying (4) by $T_n'(x)$, (11) by $K_n'(x)$ and adding, on some adjustments it is obtained that

$$\int_0^x \{ K_n'(x)\, T_n'(x) - K_n(x)\, T_n(x) \}\, dx = x \{ K_n'(x)\, T_n'(x) - K_n(x)\, T_n'(x)\} + K_n(x)\, \{nT_n(x) + \tfrac{2}{\pi}\} + \tfrac{2}{n\pi^2} \sin n\pi \tag{12}$$

From (6), on integration by parts, we get

$$\begin{aligned}\int_0^\infty x^s\, e^{-x}\, K_{2n}(x)\, dx &= 0\,; \qquad s = 0, 1, \ldots, n-2, \\ &= \frac{(-1)^n}{n\,!}\, \frac{\Gamma(s+1)}{2^{s+1}}\, \frac{\Gamma(s+2)}{\Gamma(s-n+2)}\,, \quad s \geqslant n-1.\end{aligned} \tag{13}$$

Next, for simple Laguerre polynomial $L_n(x)$, we have

$$L_n(x) = \frac{e^x}{n\,!}\, \frac{d^n}{d\,x^n}\, (x^n\, e^{-x})\,,$$

so that, on comparison with (6), it gives

$$K_{2n}(x) = (-1)^n\, \frac{xe^x}{n}\, \frac{d}{dx}\, \{e^{-2x}\, L_{n-1}(2x)\}. \tag{14}$$

Thus, we get

$$\int_0^x \frac{e^{-x}\, K_{2n}(x)}{x}\, dx = \frac{(-1)^n}{n}\, e^{-2x}\, L_{n-1}(2x) - \frac{(-1)^n}{n\,!}\,. \tag{15}$$

From the recurrence formulae[2], for $L_n(x)$, (14) becomes

$$K_{2n}(x) = (-1)^n\, \frac{xe^{-x}}{n}\, \frac{d}{dx}\, \{ L_n(2x) \}, \tag{16}$$

this gives

$$\int_0^x \frac{e^x\, K_{2n}(x)}{x}\, dx = \frac{(-1)^n}{n}\, L_n(2x) - \frac{(-1)^n}{n\,!\,n}\,. \tag{17}$$

3. The functions $K_n(x_1\, x_2)$ and $T_n(x_1, x_2)$

The functions $K_n(x)$ and $T_n(x)$ are generalised to the corresponding functions of two arguments $K_n(x_1, x_2)$ and $Tn(x_1, x_2)$ respectively, where

$$K_n(x_1, x_2) = \frac{2}{\pi}\int_0^{\pi/2} \cos\, (x_1 \tan\theta + x_2 \tan 2\theta - n\theta)\, d\theta, \tag{18}$$

$$T_n(x_1, x_2) = \frac{2}{\pi}\int_0^{\pi/2} \sin\, (x_1 \tan\theta + x_2 \tan 2\theta - n\theta)\, d\theta. \tag{19}$$

These extensions besides giving the properties of the new functions, similar to the parent functions, they also provide an association between the extended functions, as it is expressed by the various results given in this section.

(18) may also be written as

$$K_n(x_1, x_2) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} e^{i(x_1 \tan\theta + x_2 \tan 2\theta - n\theta)}\, d\theta. \tag{20}$$

Obviously

$$K_n(x,0) = K_n(x)\ ;\ T_n(x,0) = T_n(x). \tag{21}$$

And when n is any other integer or zero, we have

$$K_{4n}(0,x) = K_{2n}(x)\ ;\ K_{4n+1}(0,x) = \tfrac{1}{2} K_{\frac{1}{2}(4n+1)}(x) - \tfrac{1}{2} T_{\frac{1}{2}(4n+1)}(x), \tag{22}$$

$$K_{4n+2}(0,x) = 0\ ;\ K_{4n+3}(0,x) = \tfrac{1}{2} K_{\frac{1}{2}(4n+3)}(x) + \tfrac{1}{2} T_{\frac{1}{2}(4n+3)}(x) \tag{23}$$

Similar results may also be given for the allied function.

For brevity in the following treatments let us write K_n and T_n in place of $K_n(x_1, x_2)$ and $T_n(x_1, x_2)$ respectively. So that from (18), we have

$$\frac{\partial}{\partial x_r}\,(K_{n-r} + K_{n+r}) = K_{n-r} - K_{n+r}, \tag{24}$$

where $r = 1, 2$. In this K may be replaced by T throughout.

From (24) a number of interesting identities may be obtained.

To derive the pure recurrenece relation satisfied by K_n if we consider the identity

$$\frac{2}{\pi}\int_0^{\pi/2} \frac{d}{d\theta}\,\{\cos^2\theta \cos^2 2\theta \sin(x_1 \tan\theta + x_2 \tan 2\theta - n\theta)\}\, d\theta = 0,$$

after a bit of simplifications, we find that

$$(6-n)\,K_{n-6} - (6+n)\,K_{n+6} + 2(4+2x_1-n)\,K_{n-4} - 2(4-2x_1+n)\,K_{n+4}$$
$$+ (6+8x_2-3n)\,K_{n-2} - (6-8x_2+3n)\,K_{n+2} + 4\,(2x_1+4x_2-n)\,K_n = 0. \tag{25}$$

Similarly

$$\frac{2}{\pi}\int_0^{\pi/2} \frac{d}{d\theta}\,\{\cos^2\theta \cos^2 2\theta \cos(x_1 \tan\theta + x_2 \tan 2\theta - n\theta)\}\, d\theta = -\frac{2}{\pi},$$

gives

$$(6-n)\,T_{n-6} - (6+n)\,T_{n+6} + 2\,(4+2x_1-n)\,T_{n-4} - 2(4-2x_1+n)\,T_{n+4}$$
$$+ (6+8x_2-3n)\,T_{n-2} - (6-8x_2+3n)\,T_{n+2} + 4\,(2x_1+4x_2-n)\,T_n = 0. \tag{26}$$

It is interesting to note that the expressions like $\sum_{n=1}^{\infty} \frac{K_n}{n}$ and $\sum_{n=1}^{\infty} \frac{T_n}{n}$ reduces to certain simple difinite integrals. To obtain integral representation for $\sum_{n=1}^{\infty} \frac{K_n}{n}$, from (18),

we have

$$\sum_{n=1}^{\infty} \frac{K_n}{n} = \frac{2}{\pi} \sum_{n=1}^{\infty} \int_0^{\pi/2} \cos (x_1 \tan \theta + x_2 \tan 2\theta) \frac{\cos n\theta}{n} d\theta$$

$$+ \frac{2}{\pi} \sum_{n=1}^{\infty} \int_0^{\pi/2} \sin (x_1 \tan \theta + x_2 \tan 2\theta) \frac{\sin n\theta}{n} d\theta.$$

Now, to change the order of summation and integration, we note that $\sum_{n=1}^{\infty} \frac{\cos n\theta}{n}$ and $\sum_{n=1}^{\infty} \frac{\sin n\theta}{n}$ are non-uniformly convergent near $\theta = 0$, but on using[4] Abel's theorem, we have

$$\sum_{n=1}^{\infty} \frac{K_n}{n} = \lim_{\alpha \to 1-0} \frac{2}{\pi} \sum_{n=1}^{\infty} \int_0^{\pi/2} \cos (x_1 \tan \theta + x_2 \tan 2\theta) \frac{\alpha^n \cos n\theta}{n} d\theta$$

$$+ \lim_{\alpha \to 1-0} \frac{2}{\pi} \sum_{n=1}^{\infty} \int_0^{\pi/2} \sin (x_1 \tan \theta + x_2 \tan 2\theta) \frac{\alpha^n \sin n\theta}{n} d\theta.$$

Since α is less than 1, therefore, $\sum_{n=1}^{\infty} \frac{\alpha^n \cos n\theta}{n}$ and $\sum_{n=1}^{\infty} \frac{\alpha^n \sin n\theta}{n}$ converge uniformly throughout the range of integration, and so, the interchange of the summation and integration is now permissible. Hence after summing up under the sign of integration and taking limit, we get

$$\sum_{n=1}^{\infty} \frac{K_n}{n} = -\frac{2}{\pi} \int_0^{\pi/2} \cos (x_1 \tan \theta + x_2 \tan 2\theta) \log (2 \sin \frac{\theta}{2}) d\theta$$

$$+ \frac{1}{\pi} \int_0^{\pi/2} \sin (x_1 \tan \theta + x_2 \tan 2\theta). (\pi - \theta) d\theta. \qquad (27)$$

Similarly

$$\sum_{n=1}^{\infty} \frac{T_n}{n} = -\frac{2}{\pi} \int_0^{\pi/2} \sin (x_1 \tan \theta + x_2 \tan 2\theta) \log (2 \sin \frac{\theta}{2}) d\theta$$

$$- \frac{1}{\pi} \int_0^{\pi/2} \cos (x_1 \tan \theta + x_2 \tan 2\theta) (\pi - \theta) d\theta. \qquad (28)$$

Following the above procedure it is found that

$$\sum_{n=0}^{\infty} \frac{K_{2n+1} - K_{-2n-1}}{2n+1} = \frac{\pi}{2} T_0 ; \quad \sum_{n=0}^{\infty} \frac{T_{2n+1} - T_{-2n-1}}{2n+1} = \frac{\pi}{2} K_0 . \qquad (29)$$

We shall also evaluate $\int_0^{\infty} e^{-x_1} K_n (x_1 + a_1, x_2) dx_1$ and $\int_0^{\infty} e^{-x_2} K_n (x_1, x_2 + a_2) dx_2$ and similar integrals for the allied function T_n. For

$$\int_0^{\infty} e^{-x_1} K_n (x_1 + a_1, x_2) dx_1 = \frac{2}{\pi} \int_0^{\infty} e^{-x_1} dx_1$$

$$\times \int_0^{\pi/2} \cos \{ (x_1 + a_1) \tan \theta + x_2 \tan 2\theta - n\theta \} d\theta,$$

on changing the order of integration (by virtue of the absolute convergence of the double integral), on integration, we get

$$\int_0^\infty e^{-x_1} K_n(x_1 + a_1, x_2)\, dx_1 = \tfrac{1}{2} \{ K_{n-2}(a_1, x_2) + K_n(a_1, x_2) \}, \tag{30}$$

Similarly, it is found that

$$\int_0^\infty e^{-x_2} K_n(x_1, x_2 + a_2)\, dx_2 = \tfrac{1}{2} \{ K_{n-4}(x_1, a_2) + K_n(x_1, a_2) \}. \tag{31}$$

In (30) and (31) K may be replaced by T throughout.

The functions K_n and T_n studied in this section may be extended to the functions of m arguments $x_1, \ldots, x_m$ by

$$K_n(x_1, \ldots, x_m) = \frac{2}{\pi}\int_0^{\pi/2} \cos (x_1 \tan \theta + \ldots + x_m \tan m\theta - n\theta)\, d\theta, \tag{32}$$

and

$$T_n(x_1, \ldots, x_m) = \frac{2}{\pi}\int_0^{\pi/2} \sin (x_1 \tan \theta + \ldots + x_m \tan m\theta - n\theta)\, d\theta \tag{33}$$

respectively, and, the properties of these functions may be given in a similar manner.

REFERENCES

1. Bateman, H. *Trans. America. Soc.*, 33 : 817–31, (1931).
2. Rainville, E. D. Special Functions. *Macmillan*, 213–214 (1960).
3. Srivastava, H. M. *Bull Cal. Math. Soc*, 42 : 82–87, (1950).
4. Watson, G. N. Theory of Bessel Functions. *Cambridge*, 68, (1958).

ON ABSOLUTE CESÀRO SUMMABILITY OF ORTHOGONAL SERIES

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ABSTRACT

1. Let the Fourier expansion of a function $f(x) \varepsilon L^2(a, b)$ with respect to an orthonormal system of functions $\{\phi_n(x)\}$ $(n=0, 1, 2, \ldots)$ is given by

$$(1{\cdot}1) \qquad f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x),$$

where

$$a_n = \int_a^b f(x)\, \phi_n(x)\, dx.$$

Considering,

$$A_m = (a^2_{2^m+1} + a^2_{2^m+2} + \ldots + a^2_{2^{m+1}})^{\frac{1}{2}}$$
for $m=0, 1, 2, \ldots,$

Tandori has proved that if

$$\sum_{m=0}^{\infty} A_m < \infty$$

then (1·1) is $|C, 1|$ summable almost everywhere in (a, b). In this paper we have generalised this result and proved the following theorem ;

Theorem : If

$$(1{\cdot}2) \qquad \left\{ \sum_{m=0}^{\infty} A_m + \sum_{m=1}^{\infty} \frac{|a_m|}{m^{\alpha}} \right\} < \infty$$

then the orthogonal series (1·1) *is summable* $|C, \alpha|$ *almost everywhere in* (a, b) *provided* $\alpha > \frac{1}{2}$.

INTRODUCTION

1. The Fourier expansion corresponding to a function $f(x) \varepsilon L^2(\alpha, b)$ in terms of an orthonormal system $\{\phi_n(x)\}$ is given by

$$(1{\cdot}1) \qquad f(x) \simeq \sum_{n=0}^{\infty} a_n \phi_n(x),$$

where

$$a_n = \int_a^b f(x)\, \phi_n(x)\, dx.$$

The series (1·1) is said to be $|C, \alpha|$ - summable if

$$\sum_{n=1}^{\infty} |\sigma^{\alpha}{}_{n+1}(x) - \sigma^{\alpha}{}_{n}(x)| < \infty,$$

where

$\sigma^{\alpha}{}_{n}(x)$ designates the nth Cesaro mean of order α. Let us consider

$$A_m = (a^2{}_{2^{m}+1} + \ldots + a^2{}_{2^{m+1}})^{\frac{1}{2}}$$

$$\text{for } m = 0, 1, 2, \ldots$$

The almost everywhere $| C, 1 |$ – summability of the series (1·1) has been studied by Kåroly Tandori.[1] The object of this note is to prove the following generalisation of Tandori's theorem

Theorem. If

$$\left\{ \sum_{m=0}^{\infty} A_m + \sum_{m=1}^{\infty} \frac{| a_m |}{m^{\alpha}} \right\} < \infty,$$

then the orthogonal expansion (1·1) *is summable* $| C, \alpha |$ *almost everywhere in the range* (a, b), *provided* $\alpha < \frac{1}{2}$.

2. *Proof of the theorem*

Without restricting the generality, we assume $a_0 = a_1 = 0$.
Hence

$$\sigma^{\alpha}{}_{n+1}(x) - \sigma^{\alpha}{}_{n}(x) = \frac{1}{A^{\alpha}{}_{n+1}} \sum_{k=2}^{n+1} A^{\alpha}{}_{n-k+1}\, a_k\, \phi_k(x) - \frac{1}{A^{\alpha}{}_{n}} \sum_{k=2}^{n} A^{\alpha}{}_{n-k}\, a_k\, \phi_k(x)$$

$$= \sum_{k=2}^{n} \left(\frac{A^{\alpha}{}_{n-k+1}}{A^{\alpha}{}_{n+1}} - \frac{A^{\alpha}{}_{n-k}}{A^{\alpha}{}_{n}} \right) a_k\, \phi_k(x) + \frac{1}{A^{\alpha}{}_{n+1}}\, a_{n+1}\, \phi_{n+1}(x)$$

$$= \sum_{k=2}^{n} \frac{A^{\alpha}{}_{n-k}}{A^{\alpha}{}_{n+1}} \frac{\alpha k\, a_k\, \phi_k(x)}{(n+1)\,(n-k+1)} + \frac{1}{A^{\alpha}{}_{n+1}}\, a_{n+1}\, \phi_{n+1}(x).$$

By Schwarz's inequality we have

$$\sum_{n=1}^{\infty} \int_a^b | \sigma^{\alpha}{}_{n+1}(x) - \sigma^{\alpha}{}_{n}(x) |\, dx = O\,(1) \sum_{n=1}^{\infty} \left\{ \sum_{k=2}^{n} \left(\frac{A^{\alpha}{}_{n-k}}{A^{\alpha}{}_{n+1}} \frac{\alpha k}{(n+1)\,(n-k+1)} \right)^2 a^2{}_k + \left(\frac{1}{A^{\alpha}{}_{n+1}} \right)^2 a^2{}_{n+1} \right\}^{\frac{1}{2}}$$

$$= O\,(1) \sum_{n=1}^{\infty} \left\{ \sum_{k=2}^{n+1} \frac{k^2 (n-k+1)^{2(\alpha-1)}}{n^{2(\alpha+1)}}\, a^2{}_k \right\}^{\frac{1}{2}} + O\,(1) \sum_{n=1}^{\infty} \frac{| a_n |}{n^{\alpha}}$$

$$= O\,(1) \sum_{n=1}^{\infty} \left\{ \sum_{k=2}^{n+1} \frac{1}{n^4}\, k^2\, a^2{}_k \right\}^{\frac{1}{2}} + O\,(1) \sum_{n=1}^{\infty} \frac{| a_n |}{n^{\alpha}}$$

$$= O\,(1) \sum_{n=1}^{\infty} \frac{1}{n^2} \left\{ \sum_{k=2}^{n+1} k^2\, a^2{}_k \right\}^{\frac{1}{2}} + O\,(1) \sum_{n=1}^{\infty} \frac{| a_n |}{n^{\alpha}}$$

$$= O\,(1) \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{m=0}^{[\log\,(n+1)]} 2^{m+1} A_m + O(1) \sum_{n=1}^{\infty} \frac{| a_n |}{n^{\alpha}}$$

$$= O(1) \sum_{m=0}^{\infty} A_m \, 2^{m+1} \sum_{\substack{n \\ \log(n+1) \geqslant m}} \frac{1}{n^2} + O(1) \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\alpha}}$$

$$= O(1) \sum_{m=0}^{\infty} A_m + O(1) \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\alpha}}$$

$$< \infty \, .$$

Hence the theorem follows by an application of Levy's theorem.

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REFERENCE

1. Karoly Tandori. On the orthogonal functions (Absolute Summation), *Acta Sci. Math. Szeged.*, 21 : 292–299, (1960).

ON THE OSCILLATIONS OF A PENDULUM OF A VARIABLE LENGTH AND A PENDULUM UNDER PARAMETRIC EXCITATION

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ABSTRACT

In this paper three objects have been fulfilled. Firstly an improved result for the oscillations of a pendulum with increasing length has been obtained by using Gegenbauer polynomial approximation. Secondly the oscillations of a pendulum whose length is decreasing at a certain exponential rate have been investigated. Thirdly an approximate solution of the nonlinear differential equation depicting the motion of a simple pendulum which is excited parametrically by small vibrations of its support, has been obtained, by using Denman's linear approximation and WKBJ approximation.

1. INTRODUCTION

The differential equation for θ, the angular displacement at any time $t > 0$, of a simple pendulum whose length changes at a certain time rate is given by [6, p. 41].

$$\ddot{\theta} + \frac{2\,\dot{l}\,\dot{\theta}}{l} + \frac{g}{l}\sin\theta = 0 \tag{1·1}$$

Now by virtue of $\sin\theta$ (1·1) is a nonlinear equation. If θ is small enough, $\sin\theta \simeq \theta$ and (1·1) may be expressed in a linear form, the solution of which can be obtained in terms of Bessel functions [6, p. 41]. An application of a pendulum of varying length is found in an overhead crane [6, p. 43].

The nonlinear equation

$$\frac{d^2\theta}{dt^2} + \left(w_0^2 - \frac{\xi_0}{L} - w^2\cos wt\right)\sin\theta = 0 \tag{1·2}$$

where w, w_0, ξ_0 and L are positive constants with ξ_0/L small, depicts the motion of a simple pendulum which is excited parametrically by small vibrations of its support [9, 10].

In recent papers [2, 3, 4, 5] the linearisation of the nonlinear ordinary differential equations has been accomplished by approximating the nonlinear torque by ultraspherical polynomials or Gegenbauer polynomials.

The object of this paper is three fold. Firstly we improve upon the previous result [6, p. 42] by using Gegenbauer polynomial approximation for $\sin\theta$ instead of assuming $\sin\theta$ equal to θ. Secondly we study the oscillations of a simple pendulum when the length of the simple pendulum is decreasing, *i.e.*, when $l = l_0\, e^{-2t}$. Thirdly we obtain an approximate solution of (1·2) by using Denman's linear approximation and WKBJ approximation [1, p 253]. With these approximations the differential equation (1·2) reduces to that of the resonant circuit with varying capacitance [1, p. 254], and the salution obtained in this case may immediately be adopted.

2. GEGENBAUER POLYNOMIALS

The Gegenbauer polynomials $C_n^{\nu}(x)$ on the interval $(-1, 1)$ are the sets of polynomials orthogonal on this interval with respect to the weight factor $(1-x^2)^{\nu-\frac{1}{2}}$, each set corresponding to a value ν. They may be obtained from [8, p. 276].

$$(2{\cdot}1) \qquad (1-2xt+t^2)^{-\nu} = \sum_{n=0}^{\infty} C_n^{\nu}(x)\, t^n$$

Gegenbauer polynominals include the Legendre polynomials for which $\nu = \frac{1}{2}$.

On the interval $(-A, A)$, the Gegenbauer polynomials are defined as the sets of polynomials orthogonal on this interval with respect to the weight factor $(1-x^2/A^2)^{\nu-\frac{1}{2}}$. This gives rise to the polynomial $C_n^{\nu}(x/A)$

3. THE WKBJ APPROXIMATION

There are a number of physical problems in which the varying coefficient executes only relatively small changes about a large mean value. If the system can be described by a second order equation, this equation can be put in the form [1, p. 253]

$$(3{\cdot}1) \qquad \frac{d^2x}{dt^2} + [G(t)]^2 x = 0$$

where $G^2(t)$ includes the varying coefficient. Function $G(t)$ has a relatively large mean value about which small variations take place. If

$$|G|^2 >> \left| \frac{\ddot{G}}{2G} - \frac{3}{4}\left(\frac{\dot{G}}{G}\right)^2 \right|$$

then the solution of (6·2) may be written as

$$(3{\cdot}2) \qquad x = [G(t)]^{-\frac{1}{2}} [A \cos \phi(t) + B \sin \phi(t)]$$

where $\qquad \phi(t) = \int G(t)\, dt$

4. APPLICATION OF THE GEGENBAUER POLYNOMIALS TO THE EQUATION (1·1) ABOVE

Approximating $\sin\theta$ on the interval $(-A, A)$ with the Gegenbauer polynomials linear in θ, one obtains

$$(4{\cdot}1) \qquad \sin^* \theta = a_1^{\nu} C_1^{\nu}(\theta/A)$$

where

$$a_1^{\nu} = \frac{\int_{-A}^{A} \left(1-\frac{\theta^2}{A^2}\right)^{\nu-\frac{1}{2}} C_1^{\nu}(\theta/A) \sin\theta\, d\theta}{\int_{-A}^{A} \left[C_1^{\nu}(\theta/A)\right]^2 \left(1-\frac{\theta^2}{A^2}\right)^{\nu-\frac{1}{2}} d\theta}$$

$$= \frac{2\nu \int_{-1}^{1} (1-x^2)^{\nu-\frac{1}{2}} x \sin Ax\, dx}{\int_{-1}^{1} \left[C_1^{\nu}(x)\right]^2 (1-x^2)^{\nu-\frac{1}{2}} dx}$$

Now

$$\text{numerator} = \frac{2\nu\, \Gamma(\tfrac{1}{2})\, \Gamma(\nu + \tfrac{1}{2})\, J_{\nu+1}(A)}{(A/2)^{\nu}}$$

and

$$\text{denominator} = \frac{2\nu\, \Gamma(\tfrac{1}{2})\, \Gamma(\nu + \tfrac{1}{2})}{(\nu + 1)\, \Gamma(\nu)}$$

Hence

$$a_1^{\nu} = \frac{(\nu + 1)\, \Gamma(\nu)}{(A/2)^{\nu}}\, J_{\nu+1}(A) \tag{4·2}$$

and

$$\sin^* \theta = \frac{\Gamma(\nu + 2)}{(A/2)^{\nu+1}}\, J_{\nu+1}(A)\, \theta \tag{4·3}$$

For $0 < A \leqq 3$ radians it has been found in [2] that the value of ν corresponding to minimum maximum (minimax) error is approximately $-0{\cdot}21$.

5. SOLUTION OF (1·1)

Substitution of (4·3) in (1·1) gives

$$\ddot{\theta} + \frac{2\,\dot{l}}{l}\,\dot{\theta} + \frac{g\,\Gamma(\nu + 2)}{l\,(A/2)^{\nu+1}}\, J_{\nu+1}(A)\,\theta = 0 \tag{5·1}$$

In [6, p. 41] the problem considered is that of a simple pendulum whose length increases at a constant rate a, so that $l = l_0 + at$. Substituting this value in (5·1) we get

$$\ddot{\theta} + \left(\frac{2a}{l_0 + at}\right)\dot{\theta} + \frac{g\,\Gamma(\nu + 2)\, J_{\nu+1}(A)}{(A/2)^{\nu+1}\,(l_0 + at)}\,\theta = 0 \tag{5·2}$$

Writing
$$z = \frac{l_0 + at}{a}$$

(5·2) becomes

$$\frac{d^2\theta}{dz^2} + \frac{2}{z}\frac{d\theta}{dz} + k^2\,\frac{\theta}{z} = 0 \tag{5·3}$$

where

$$k^2 = \frac{g}{a}\;\frac{\Gamma(\nu + 2)}{(A/2)^{\nu+1}}\; J_{\nu+1}(A) \tag{5·4}$$

We observe that in (5·4) k^2 is amplitude dependent and is no longer a constant, as it is in [6].

The solution of (5·3) will be

$$\theta = z^{-\frac{1}{2}}\,[A\, J_1(2\,k\,z^{\frac{1}{2}}) + B\, Y_1(2\,k\,z^{\frac{1}{2}})] \tag{5·5}$$

If the prescribed initial conditions are

$\theta = A$ and $\dot{\theta} = 0$, when $t = 0$, then

$$\theta = \frac{A\,\pi\, g^{\frac{1}{2}}\, l_0\, k}{a\,(l_0 + at)^{\frac{1}{2}}}\,[J_2(c)\, Y_1(w) - Y_2(c)\, J_1(w)] \tag{5·6}$$

where

$$w = \frac{2\, g^{\frac{1}{2}}\,(l_0 + at)^{\frac{1}{2}}}{a} \left\{\frac{\Gamma(\nu + 2)}{(A/2)^{\nu+1}}\, J_{\nu+1}(A)\right\}^{\frac{1}{2}}$$

and

$$C = 2\, k \left(\frac{l_0}{a}\right)^{\frac{1}{2}}$$

$$= \frac{2}{a}\left[\frac{g\, l_0\, \Gamma(\nu + 2)}{(A/2)^{\nu+1}}\, J_{\nu+1}(A)\right]^{\frac{1}{2}}$$

6. PENDULUM WITH DECREASING LENGTH

Le $l = l_0\, e^{-2t}$, then (1·1) becomes

(6·1) $$\ddot{\theta} - 4\dot{\theta} + \frac{g}{l_0} e^{2t} \sin\theta = 0$$

Making use of (4·3), (6·1) becomes

(6·2) $$\ddot{\theta} - 4\dot{\theta} + \lambda^2 e^{2t}\,\theta = 0$$

where $$\lambda^2 = \frac{g\,\Gamma(\nu + 2)}{l_0\,(A/2)^{\nu+1}}\, J_{\nu+1}(A).$$

Substituting $\theta = y\, e^{2t}$ in (6·2), we get

(6·3) $$\frac{d^2 y}{dt^2} + (\lambda^2 e^{2t} - 4)\, y = 0.$$

The solution of (6·3) is [7, p. 355]

(6·4) $$y = C_1 J_2(\lambda\, e^t) + C_2 Y_2(\lambda\, e^t).$$

Hence

(6·5) $$\theta = e^{2t}\,[C_1 J_2(\lambda\, e^t) + C_2 Y_2(\lambda\, e^t)]$$

Let the initial conditions be $\theta = \theta_0$, $\dot{\theta} = 0$ when $t = 0$. Now since

$$J_1(c)\, Y_2(c) - J_2(c)\, Y_1(c) = -\frac{2}{\pi c}$$

the application of initial conditions will give

(6·6) $$\theta = \frac{\pi\,\lambda\,\theta_0\, e^{2t}}{2}\,[Y_1(\lambda) J_2(\lambda\, e^t) - J_1(\lambda)\, Y_2(\lambda\, e^t)].$$

7. SOLUTION OF (1·2)

Applying (4·3) to (1·2) we obtain

(7·1) $$\ddot{\theta} + \left(1 - \frac{\xi_0}{L}\frac{w^2}{w_0^2}\cos wt\right)\frac{\Gamma(\nu + 2)}{(A/2)^{\nu+1}}\, J_{\nu+1}(A)\,\theta = 0$$

Let

(7·2) $$0 < \frac{\xi_0}{L}\frac{w^2}{w_0^2} << 1,$$

(7·3) $$w^{*2} = \frac{\Gamma(\nu + 2)}{(A/2)^{\nu+1}}\, w_0^2\, J_{\nu+1}(A)$$

and

$$m = \frac{\xi_0 \, w^2}{w_0^2 \, L}$$

then the solution of (7·1) will be [1, p. 254].

$$\theta = w^{* -\frac{1}{2}} \left(1 + \frac{m}{4} \cos wt\right)$$

$$\times \left[A \cos w^* \left(t - \frac{m}{2w} \sin wt \right) + B \sin w^* \left(t - \frac{m}{2w} \sin wt \right) \right] \tag{7·4}$$

REFERENCES

1. Cunningham, W. J. Introduction to non-linear analysis. *McGraw-Hill*, (1958).
2. Denman, H. H. and Howard, J E. Applications of ultraspherical polynomials to non-linear oscillations. I. Free oscillation of the pendulum. *Q. Appl. Math*, 21: 325–330, (1964).
3. Denman, H. H. and Liu, Y. K. Applications of ultraspherical polynomials to nonlinear oscillations. II. Free oscillations, *Q. Appl. Math.*, 22 : 273–292, (1964).
4. Garde, R. M. Applications of Gegenbauer polynomials to nonlinear oscillations—Forced and free oscillations without damping. (to be published in the Indian Journal of Mathematics).
5. Garde, R. M. Application of Gegenbaur polynomials to nonlinear damped oscillations. (communicated for publication to the Indian Jonrnal of Mathematics).
6. McLachlan, N. W. Bessel fnnctions for engineers. *Oxford*. 2nd Ed., (1961).
7. Pipes, L. A Applied Mathematics for engineers and physicists. *McGraw-Hill*, 2nd Ed.
8 Rainville, E. D. Special functions. *Macmillan Co., New York*, (1960).
9. Struble, R. A. Oscillations of a pendulum under parametric excitation. *Q. App. Math.*, 21 : 121-131, (1963).
10. Struble, R. A. On the oscillations of a pendulum under parametric excitation. *Q. Appl. Math.* 22 : 157–159, (1964)

OPERATIONAL FORMULAE FOR THE SOLUTIONS OF F-EQUATION

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1·1. ABSTRACT

Burchnall (1941), Carlitz (1960) and Chatterjee (1963) have employed the operational formulae for Hermite laguerre and other polynomials. In this paper we have, deduced the operational formulae for the solutions of F-equation, which are of hypergeometric character. The results of the above authors have been obtained as particular cases of the results of this paper.

1·2. We shall prove the following formulae :

$$\sum_{r=0}^{n} \binom{n}{r} F(z,\alpha+n-r)\, D^r f(z)$$

$$= z^{-n} F(z,\alpha) \prod_{j=1}^{n} \left[z D + z \frac{F(z,\alpha+1)}{F(z,\alpha)} - j + 1 \right] f(z) \tag{1}$$

$$= F(z,\alpha) \left[D + \frac{F(z,\alpha+1)}{F(z,\alpha)} \right]^n f(z) \tag{2}$$

and

$$\sum_{r=0}^{n} \binom{n}{r} y(z,\alpha-n+r)\, D^r f(z)$$

$$= z^{-n} y(z,\alpha) \prod_{j=1}^{n} \left[z D + z \frac{y(z,\alpha-1)}{y(z,\alpha)} - j + 1 \right] f(z) \tag{3}$$

$$= y(z,\alpha) \left[D + \frac{y(z,\alpha-1)}{y(z,\alpha)} \right] f(z) \tag{4}$$

Proof :

Consider,

$${}_{\Omega_{-n}} f(z) = D^n \left[F(z,\alpha) f(z) \right]$$

where $F(z,\alpha)$ satisfies the F-equation [9, p. 15]

$$\frac{\partial F(z,\alpha)}{\partial z} = F(z,\alpha+1)$$

and $f(z)$ is any arbitrary functian of z. By expansion of the right side we get,

$${}_{\Omega_{-n}} f(z) = \sum_{r=0}^{n} \binom{n}{r} F(z,\alpha+n-r)\, D^r f(z) \tag{A}$$

where

$$D = \frac{d}{dz}$$

Now using the known results [*see* 5, p. 164]

(5) $$\begin{cases} z^n D^n f(z) = \prod_{j=0}^{n-1} (\delta - i) f(z) \\ \text{and} \\ D^n [e^{\phi(z)} f(z)] = e^{\phi(z)} [D + \phi^1(z)]^n f(z) \end{cases}$$

where $$\delta = z \frac{\partial}{\partial z}$$

we obtain,

$$z^n \,{}_{-}\Omega_{-n}\; f(z) = \prod_{j=1}^{n} [z D - j + 1]\, e^{\log F(z,\alpha)}$$

(B) $$= F(z,\alpha) \prod_{j=1}^{n} \left[z D + z \frac{F(z,\alpha+1)}{F(z,\alpha)} - j + 1 \right] f(z)$$

and

$$\cdot\,\Omega_{-n}\; f(z) = D^n [\exp (\log F(z,\alpha)\, f(z)]$$

(C) $$= F(z,\alpha) \left[D + \frac{F(z,\alpha+1)}{F(z,\alpha)} \right]^n f(z)$$

Hence on equating (*A*), (*B*), and (*C*) we get (1) and (2). Similarly we can obtain (3) and (4), on considering

$$-\Omega_{-n}^{1}\, f(z) = D^n [\,y(z,\alpha) f(z)]$$

where $y(z,\alpha)$ satisfies the equation[1]

$$\frac{\partial}{\partial z} y(z,\alpha) = y(z,\alpha - 1).$$

Again using [7, p. 90]

$$y(z,\alpha) = \phi\,(z,\alpha)\, F(z,\alpha)$$

where

$$\phi(z,\alpha) = q(\alpha) \exp\,[p_0(z) + \alpha\, p_1(z)]$$

From (5) we get,

$$\sum_{r=0}^{n} \binom{n}{r} y(z,\alpha - n + r)\; D^r f(z)$$

(6) $$= \sum_{r=0}^{n} \binom{n}{r} F(z,\alpha + n - r)\, \phi(z,\alpha) \left[D + \frac{\phi^1(z,\alpha)}{\phi(z,\alpha)} \right]^r f(z)$$

(7) $$= y(z,\alpha) \left[D + \frac{\phi^1(z,\alpha)}{\phi(z,\alpha)} + \frac{F(z,\alpha+1)}{F(z,\alpha)} \right]^n f(z)$$

1·3. *Special cases.*

(*i*) when $f(z) = 1$ then from (2) we get,

$$F(z,\alpha + n) = F(z,\alpha) \left[D + \frac{F(z,\alpha+1)}{F(z,\alpha)} \right]^n 1$$

hence

$$F(z,\alpha+m+n) = F(z,\alpha)\left[D+\frac{F(z,\alpha+1)}{F(z,\alpha)}\right]^n \left[\frac{F(z,\alpha+m)}{F(z,\alpha)}\right]$$

(8) $$= \sum_{r=0}^{n} \binom{n}{r} F(z,\alpha+n-r)\, D^r \left[\frac{F(z,\alpha+m)}{F(z,\alpha)}\right]$$

(*ii*) If we substitute $f(z) = 1$ in (1) we get,

$$F(z,\alpha+n) = \frac{F(z,\alpha)}{z^{n-1}}\, D\left[\frac{z^{n-1}F(z,\alpha+n-1)}{F(z,\alpha)}\right] + \frac{F(z,\alpha+1)\,F(z,\alpha+n-1)}{F(z,\alpha)}$$

(9) $$-(n-1)\,\frac{F(z,\alpha+n-1)}{z}$$

1·4. *Particular cases*

(*i*) Let [9, p. 21]

$$F(z,\alpha) = \exp(-z^2)\, H_\alpha(-z)$$

then from (2) when $\alpha = 0$, we obtain [2, p. 10]

$$\sum_{r=0}^{n} \binom{n}{r} (-)^{n-r} H_{n-r}(z)\, D^r f(z) = [D-2z]^n f(z)$$

(*ii*) Let [9, p. 21]

$$F(z,\alpha) = \Gamma(\alpha+1)\,(-z)^{-\alpha-1-\mu}\, e^{-1/z}\, L_\alpha^{b}(1/z)$$

then from (2) and (1) when $f(z) = 1$, $1/z = x$ and $\alpha = 0$ we get,

[5, p. 169]

$$L_n^{b}(x) = \frac{x^n}{n!}\left[D+\frac{b+n-x}{x}\right]^n 1$$

and

[3, p. 220]

$$L_n^{(b)}(x) = \frac{1}{n!}\prod_{j=1}^{n} [xD+j+b-x]\; 1$$

(*iii*) from (7) when $f(z) = 1$ and $n = 1$ we get [7, p. 96]

$$F(z,\alpha+1) + \frac{\phi^1(z,\alpha)}{\phi(z,\alpha)} F(z,\alpha) - \frac{\phi(z,\alpha-1)}{\phi(z,\alpha)} F(z,\alpha-1) = 0$$

(*iv*) when

$$F(z,\alpha) = \exp(-z^2)\, H_\alpha(-z)$$

then from (8) we get [8, p. 188] when $\alpha = 0$

$$H_{m+n}(z) = \sum_{r=0}^{\min(m,n)} \binom{m}{r} (-)^r H_{m-r}(z)\, H_{n-r}(z) \cdot \frac{2^r\, n!}{(n-r)!}$$

(v) Let [7, p. 95]

$$F(z,\alpha) = (-)^{\alpha}\ z^{-\alpha/2} J_{\alpha}(2\sqrt{z})$$

$$\phi(z,\alpha) = (-)^{\alpha} z^{\alpha}$$

and $$y(z,\alpha) = z^{\alpha/2} J_{\alpha}(2\sqrt{z})$$

then from (6) we get,

$$\sum_{r=0}^{n} \binom{n}{r} z^{r/2} J_{\alpha-n+r}(2\sqrt{z}) D^r f(z)$$

$$= \sum_{r=0}^{n} \binom{n}{r} (-)^{n-r} z^{r/2} J_{\alpha+n-r}(2\sqrt{z}) \left[D + \frac{\alpha}{z}\right]^r f(z)$$

In particular when $f(z) = 1$ we get,

$$J_{\alpha-n}(2\sqrt{z}) = \sum_{r=1}^{n} \binom{n}{r} (-)^{n-r} z^{r/2} J_{\alpha+n-r}(2\sqrt{z}) \left[D + \frac{\alpha}{z}\right]^r 1$$

when $\alpha = 0$ it reduces to well known transformation,

$$J_{-n}(2\sqrt{z}) = (-)^n J_n(2\sqrt{z})$$

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REFERENCES

1. Agrawal, B. M. On deceeding F-equaticn. *To be published in the J. Vikram* 4 (1966).
2. Burchnall, J. L. *Quart J. Math. (Oxford)* 9-11, (1941).
3. Carlitz, L. *Mich. Math. J.* 219-223, (1960).
4. Chatterjee, S. K. *Quart. J. Math. (Oxford)* (1963).
5. Chatterjee, S. K. *Rend. Sem. Math. Padova* 163-169, (1963).
6. Chatterjee, S. K. *ibid.* 271-277, (1963).
7. Chako, N. and Meixner, L *Arch. Ratio. Mech. Analy.* 89-96. (1959).
8. Rainville E. D. *Special Functions.* (1960).
9. Truesdell, C. *A Unified Theory of Special Functions*, (1948).

AN INTEGRAL EQUATION INVOLVING JACOBI POLYNOMIAL

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ABSTRACT

In this paper we solve an integral equation involving Jacobi Polynomial by the application of Mellin transforms. We also discuss a particular case.

1. INTRODUCTION

Inversion integrals for integral equations involving Tchebicheff, Legendre and Gegenbauer polynomials respectively were given by[1,2,3]. Inversion integrals with other polynomials and functions in the kernel have also been obtained. Bushman[3] used Mellin transforms in deriving his results. By applying the same method as used by[3], Srivastava[4] has recently shown if

(*i*) k and n are integers with $0 < k < n$; and $-1 < \beta < 1$

(*ii*) $f^m(1) = 0, f_1^m(1) = 0$, for $0 \leqslant m \leqslant 2k$

and $f^{2k+1}(x), f_1^{2k+1}(x)$ are piecewise continuous for $0 < a \leqslant x < 1$ then the integral equations

$$(1\cdot1) \qquad \int_x^1 F_n^{(k,\beta)}\left(\frac{t}{x}\right) g(t)\,dt = f(x)$$

$$(1\cdot2) \qquad \int_x^1 R_n^{(k,\beta)}\left(\frac{t}{x}\right) g_1(t)\,dt = f_1(x)$$

have the solutions

$$(1\cdot3) \qquad g(t) = \int_t^1 G_n^{(k,\beta)}\left(\frac{t}{y}\right) y^{-2n+2k+2\beta-1} \times \left(-y^{-1}\frac{d}{dy}\right)^{2k+1}\left[y^{2n+2k-2\beta+1} f(y)\right] dy$$

$$(1\cdot4) \qquad g_1(t) = \int_t^1 S_n^{(k,\beta)}\left(\frac{t}{y}\right) y^{-2n+2k-2\beta+2} \times \left(-y^{-1}\frac{d}{dy}\right)^{-1}\left[y^{2n+2k+2\beta-2} f_1(y)\right] dy$$

respectively for $0 < a \leqslant t < 1$ where

$$F_n^{(k,\beta)}(x) = \frac{(n)!\, x(x^2-1)^{k-\beta}\, P_n^{(k-\beta,\beta)}(2x^2-1)}{2^{k-\frac{1}{2}}\,\Gamma(k-b+n+1)}$$

$$G_n^{(k,\beta)}(x) = \frac{\Gamma(n-k+1)\,(1-x^2)^{k+\beta-1}\, P_{n-k}^{(k+\beta-1,-\beta)}(2x^2-1)}{2^{k-\frac{1}{2}}\,\Gamma(n+\beta)}$$

$$R_n^{(k,\beta)}(x) = \frac{(n)!\,(x^2-1)^{k+\beta-1}\ P_n^{(k+\beta-1,\,-\beta)}\ (2x^2-1)}{2^{k-\frac{1}{2}}\ \Gamma(n+k+\beta)}$$

$$S_n^{(k,\beta)}\ (x) = \frac{\Gamma(n-k)\ x(1-x^2)^{k-\beta}\ P_{n-k-1}^{(k-\beta,\beta)}\ (2x^2-1)}{2^{k-\frac{1}{2}}\ \Gamma(n-\beta)}$$

$P_N^{(A,\,B)}\ (y)$ being Jacobi polynomials.

In the present paper, the solution of an integral equation involving Jacobi polynomial has been discussed under somewhat different conditions than those of Srivastava[4]. The method adopted is that of Bushman[3].

The restrictions $\alpha > -1$, $\beta > -1$ are placed in the definition of Jacobi polynomials $P_n^{(\alpha,\,\beta)}\ (x)$ for the convergence of the integrals involving the product of Jacobi polynomials with their weight function $(1-x)^\alpha\ (1+x)^\beta$. Considered as a polynomial or defined in terms of Rodrigue formula, these restrictions seem to be unnecessary, hence we have taken the liberty to voilate the condition $\beta > -1$. It will be observed throughout that this does not affect the validity of the theorem established in the paper. We conclude with a particular case which has not been worked out previously and can not be derived from the results of others.

2. Results required in the proof.

We shall represent the Mellin transform

$$F(s) = \int_0^\infty f(x)\ x^{s-1}\ dx,\ Re\ s > 0, \text{ by}$$

(2·1) $$F(s) = M\{f(x)\ ;\ s\}$$

We have from [5, p. 307, p. 308, p. 311]

(2·2) $$(-1)^n\ (s-n)_n\ F(s-n) = M\ \{\ f^n\ (x)\ ;\ s\ \}$$

(2·3) $$F_1(s+\alpha)\ F_2(1-s-\alpha+\beta) = M\ \left\{x^\alpha \int_0^\infty \xi^\beta f_1(x\xi)\ f_2(\xi)\ d\xi\ ;\ s\ \right\}$$

(2·4) $$h^{-1}\ B(\nu,\ s/h) = M\ \{(1-x^h)^{\nu-1}\ [1-U(x-1)]\ ;\ s\}$$

$$Re\ s > 0,\ h > 0,\ Re\ \nu > 0$$

(2·5) $$h^{-1}\ B(1-\nu-s/h,\ \nu) = M\{(x^h-1)^{\nu-1}\ [U(x-1)\ ]\ ;\ s\ \}$$

$$h > 0,\ Re\ \nu > 0,\ Re\ s < h - h\ Re\ \nu$$

3. *Theorem.* If

(*i*) n and α are positive integers and β is an integer such that $\beta < n + \alpha + 1$

(*ii*) $f^{2\alpha+2}\ (x)$ is piecewise continuous for $0 < a \leqslant x < 1$ and $f(1) = f^1(1) = \ldots\ldots = f^{2\alpha+1}\ (1) = 0$

then the integral equation

(3·1) $$\int_x^1 F_n^{(\alpha,\beta)} \left(\frac{t}{x}\right) g(t)\ dt = f(x)$$

has the solution

$$g(t) = \int_t^1 G_n^{(\alpha,\beta)}\left(\frac{t}{y}\right) y^{n+2\alpha+1} \times \left(\frac{d}{dy}\right)^{2\alpha+2} \left[y^{-n-1} f(y) \right] dy \tag{3·2}$$

for $0 < a \leqslant t < 1$ where

$$F_n^{(\alpha,\beta)}(x) = \frac{(n)!\,(x-1)^{\alpha}\, x^{-\beta}\, P_n^{(\alpha,-\beta)}(2x-1)}{(\alpha+n)!}$$

$$G_n^{(\alpha,\beta)}(x) = \frac{(n+\alpha-\beta+1)!\,(1-x)^{\alpha} x^{\beta}\, P_{n+\alpha-\beta+1}^{(\alpha,\beta)}(2x-1)}{(n+2\alpha-\beta+1)!}$$

Proof:

First we shall show that for $u > 1$

$$J(u) = \int_{1/u}^{1} F_n^{(\alpha,\beta)}(uv)\; G_n^{(\alpha,\beta)}(v)\, dv = \frac{(u-1)^{2\alpha+1}\, u^{-(n+2\alpha+2)}}{(2\alpha+1)!} \tag{3·3}$$

The relation (3·3) can be written as

$$J(u) = \int_0^{\infty} F_n^{(\alpha,\beta)}(uv)\,[\,U(uv-1)\,].\; G_n^{(\alpha,\beta)}(v)\;[\,1-U(v-1)\,]\,dv \tag{3·4}$$

Applying the convolution theorem (2·3) to the relation (3·4), we get

$$M\{J(u)\,;s\} = M\{F_n^{(\alpha,\beta)}(u)\,[\,U(u-1)\,]\,;s\} \times M\{G_n^{(\alpha,\beta)}(u)\,[\,1-U(u-1)\,];\,1-s\} \tag{3·5}$$

From Rodrigue formula for Jacobi polynomial, we get

$$F_n^{(\alpha,\beta)}(x) = \frac{1}{(\alpha+n)!}\left(\frac{d}{dx}\right)^n \left[(x-1)^{\alpha+n}\, x^{-\beta+n} \right] \tag{3·6}$$

and

$$G_n^{(\alpha,\beta)}(x) = \frac{\left(\frac{d}{dx}\right)^{n+\alpha-\beta+1} \left[(1-x)^{n+2\alpha-\beta+1}\, x^{n+\alpha+1} \right]}{(-1)^{n+\alpha-\beta+1}\,(n+2\alpha-\beta+1)!} \tag{3·7}$$

Using the results (3·6), (3·7), (2·4), (2·5) and (2·2), we get

$$M\{F_n^{(\alpha,\beta)}(u)\,[\,U(u-1)\,];\,s\} = \frac{\Gamma(n-s+1)\,\Gamma(-\alpha+\beta-n-s)}{\Gamma(1-s)\,\Gamma(1+\beta-s)} \tag{3·8}$$

$$M\{G_n^{(\alpha,\beta)}(u)\,[\,1-U(u-1)\,];\,1-s\} = \frac{\Gamma(1-s)\,\Gamma(1+\beta-s)}{\Gamma(3+n+2\alpha-s)\,\Gamma(-\alpha+\beta-n-s)} \tag{3·9}$$

Hence from (3·5) we get

$$M\{J(u)\,;\,s\} = \frac{1}{(2\alpha+1)!}\,B(1+n-s,\ 2\alpha+2) \tag{3·10}$$

But from (2·5), we get

$$(3\cdot 11) \qquad \frac{B(1+n-s,\ 2\alpha+2)}{(2\alpha+1)!} = M\left\{\frac{(u-1)^{2\alpha+1}\ u^{-(n+2\alpha+2)}\ \dot{U}(u-1)}{(2\alpha+1)!}\ ;\ s\right\}$$

Hence the formula (3·3).

Now substituting the value of $g(t)$ from (3·2) into (3·1) and changing th order of integration, we get

$$(3\cdot 12) \qquad I(x) = \int_x^1 \left[\left\{ \int_x^y F_n^{(\alpha,\beta)}\left(\frac{t}{x}\right) G_n^{(\alpha,\beta)}\left(\frac{t}{y}\right) dt \right\} \times y^{n+2\alpha+1}\left(\frac{d}{dy}\right)^{2\alpha+2}\{y^{-n-1} f(y)\} \right] dy$$

If we write $v = \dfrac{t}{y}$ and $u = \dfrac{y}{x}$, the inner integral becomes $y\,J(u)$, therefore, we get

$$(3\cdot 13) \qquad I(x) = \frac{x^{n+1}}{(2\alpha+1)!}\int_x^1 (y-x)^{2\alpha+1}\left(\frac{d}{dy}\right)^{2\alpha+2}\left[y^{-n-1} f(y)\right] dy$$

Successive $(2\alpha+1)$ integrations by parts and application of the conditions (*ii*) of our theorem yield $I(x) = f(x)$.

Thus the theorem is proved.

4. *Particular case* :

If in the theorem in the section 3, we take $\alpha = \beta = 0$, we get the case of Legendre polynomial, and we have the following theorem :

If $f''(x)$ is piecewise continuous for $0 < a \leqslant x < 1$ and $f(1) = f'(1) = 0$ then the integral equation

$$(4\cdot 1) \qquad \int_x^1 P_n\left(\frac{2t}{x} - 1\right) g(t)\, dt = f(x)$$

has the solution

$$(4\cdot 2) \qquad g(t) = \int_t^1 P_{n+1}\left(\frac{2t}{y} - 1\right) y^{n+1}\left(\frac{d}{dy}\right)^2\left[y^{-n-1} f(y)\right] dy$$

$$\text{for } 0 < a \leqslant t < 1$$

The following two points should be noted :

(*i*) The argument in the above kernel is different from that of[2]

(*ii*) The above case can not be derived from the results of[3] or[4]

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REFERENCES

1. Ta-li, A new class of integral transforms, *Proc. Amer. Math. Soc.* **11** : 290–298, (1960).
2. Bushman, R. G An inversion integral for a Legendre transformation. *Amer. Math. Monthly*, **60** : 288-289, (1962).
3. Bushman, R. G. An inversion integral. *Proc. Amer. Math Soc.* **13** : 675–677, (1962).
4. Srivastava, K. N. Inversion integrals involving Jacobi polynomial. *Proc. Amer. Math. Soc* **15** (4) : 634-638, (1963).
5. Erdelyi, A. Tables of integrals transforms. Vol. 1 *McGraw-Hill, New York*, (1954).

BILINEAR GENERATING RELATIONS INVOLVING LAGUERRE POLYNOMIALS

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INTRODUCTION

1. In this paper we have estabished four bilinear generating relations involving Laguerre polynomials by using the method of series manipulations. Formula (2·1) is a generalization of Hardy-Hille formula [3, Vol. 2; p. 190], Bottema's formula[1] and formula (7·1) given by Rangarajan.[6] Formula (2·2) generalizes a result of Brafman[2]. Rangarajan's formula (7·1) can also be derived from (2·2) by a limiting process. Another formula (7·5) obtained by Rangarajan[6] can be derived by a similar limiting process from the formulae (2·3) and (2·4) which are similar in nature to (2·2). In the various formulae we have used the following definition of the Laguerre polynomial [5; p. 200]:

(1·1) $L_n^{(a)}(x) = \frac{(1+a)_n}{n!}\, {}_1F_1(-n; 1+a; x)$, where $(a)_n$ has its usual meaning.

Other results that we require in the proofs are:

(1·2) $\sum_{n=0}^{\infty} L_n^{(b-n)}(y)\, t^n = (1+t)^b e^{-yt}$, which is formula (19) in [3; p. 189],

$$(1·3) \qquad \sum_{n=0}^{\infty} \frac{(n+k)!}{n!\,k!} L_{n+k}^{(a)}(x)\, t^n = (1-t)^{-1-\alpha-k} \exp\left(\frac{-xt}{1-t}\right) L_k^{(a)}\left(\frac{x}{1-t}\right)$$

which is formula 9 in [5; p. 211].

$$(1·4) \qquad (1-t)^{-c}\, {}_1F_1\left[\begin{matrix} c; \\ 1+a; \end{matrix} \frac{-xt}{1-t}\right] = \sum_{n=0}^{\infty} \frac{(c)_n}{(1+a)_n} L_n^{(a)}(x)\, t^n,$$

which is formula (3) in [5; p. 202].

$$(1·5) \qquad {}_1F_1(a; b; x+y) = \sum_{n+0}^{\infty} \frac{(a)_n}{n!\,(b)_n}\, {}_1F_1(a+n; b+n; x)\, y^n,$$

which is formula (2·3·2) in[7]

(1·6) ${}_1F_1(a; b; x) = e^x\, {}_1F_1(b-a; b; -x)$ ——— Kummer's transformation,

which is formula (2) in [5; p. 125].

2. *Generating Relations.* We shall prove the following formulae :

$$(2\cdot1)\quad \sum_{n=0}^{\infty} \frac{n!\,(1+b+k)_n}{(1+a)_n\,(1+b)_n} L_n^{(a)}(x)\, L_n^{(b)}(y)\, t^n$$

$$= (1-t)^{-1-b-k} \exp\left[-\frac{t(x+y)}{1-t}\right] \sum_{r=0}^{\infty} \frac{(1+b+k)_r}{r!\,(1+a)_r\,(1+b)_r} \left\{\frac{xyt}{(1-t)^2}\right\}^r {}_1F_1\left[\begin{matrix} a-b-k; \\ 1+a+r; \end{matrix} \frac{xt}{1-t}\right]$$

$$\times\, {}_1F_1\left[\begin{matrix} -k; \\ 1+b+r; \end{matrix} \frac{yt}{1-t}\right].$$

$$(2\cdot2)\quad \sum_{n=0}^{\infty} {}_2F_1\left[\begin{matrix} -n,\ d; \\ 1+a; \end{matrix} x\right] L_n^{(b)}(y)\, t^n$$

$$= (1-t)^{-1-b} \exp\left(\frac{-yt}{1-t}\right) {}_3\Phi_G^{(1)}\left[d,\ d,\ d,\ c,\ a-b;\ c,\ 1+a,\ 1+a;\ \frac{-xt}{1-t}, \frac{xt}{1-t}, \frac{xyt}{(1-t)^2}\right],$$

where ${}_3\Phi_G^{(1)}(a, a, a, b, b'; c, c', c'; x, y, z) \equiv \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}\,(b)_m\,(b')_n}{m!\,n!\,p!\,(c)_m (c')_{n+p}} x^m y^n z^p$

is the confluent hypergemetric function of three variables.[4]

$$(2\cdot3)\quad \sum_{n=0}^{\infty} {}_2F_1(-n, d; 1+a; x)\ L_n^{(b-n)}(y)\, t^n = e^{-yt}(1+t)^b\, \Phi_1\left(d, -b;\ 1+a;\ \frac{xt}{1-t},\ xyt\right),$$

where $\Phi_1(a, b; c; x, y) \equiv \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m}{m!\,n!\,(c)_{m+n}} x^m y^n$

is the confluent hypergeometric function of two variables [3, vol. 1 ; p. 225].

$$(2\cdot4)\quad \sum_{n=0}^{\infty} \frac{(-1)^n(-b)_n}{(1+a)_n} L_n^{(a)}(x)\ {}_2F_1(-n, d; 1+b-n; y)\, t^n$$

$$= (1+t)^b\, {}_3\Phi_M^{(2)}\left(g, d, -b, d; g, 1+a, 1+a; yt, \frac{xt}{1-t}, xyt\right),$$

where ${}_3\Phi_M^{(2)}(a, b, b', b; c, c', c'; x, y, z) \equiv \sum_{m,n,p=0}^{\infty} \frac{(a)_m (b)_{m+p} (b')_n}{m!\,n!\,p!\,(c)_m (c')_{n+p}} x^m y^n z^p$

is the confluent hypergeometric function of three variables.[4]

3. *Proof of* (2·1). We first establish the following lemma :

$$(3\cdot1)\quad \sum_{n=0}^{\infty} \frac{(1+b+k)_{n+r}}{(1+b)_{n+r}} \frac{(n+r)!}{n!\,r!} L_{n+r}^{(b)}(y)\, t^n$$

$$= (1-t)^{-1-b-k} \exp\left(\frac{-yt}{1-t}\right) \sum_{p=0}^{r} \frac{(1+b+k)_r}{(r-p)!\, p!\, (1+b)_{r-p}} \; {}_1F_1\left[\begin{matrix} -k\,; \\ 1+b+r-p\,; \end{matrix} \; \frac{yt}{1-t}\right] \frac{(-y)^{r-p}}{(1-t)^{2r-p}}.$$

Consider the series

$$\sum_{r,n=0}^{\infty} \frac{(1+b+k)_{n+r}}{(1+b)_{n+r}} \frac{(n+r)!}{n!\, r!} L_{n+r}^{(b)}(y)\; t^n v^r$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(1+b+k)_n}{(1+b)_n} L_n^{(b)}(y) \frac{n!}{(n-r)!\, r!} t^{n-r} v^r = \sum_{n=0}^{\infty} \frac{(1+b+k)_n}{(1+b)_n}$$

$$\times L_n^{(b)}(y)\, (t+v)^n$$

$$= (1-t-v)^{-1-b-k} \; {}_1F_1\left[\begin{matrix} 1+b+k\,; \\ 1+b\,; \end{matrix} \; \frac{-y\,(t+v)}{1-t-v}\right], \text{ by } (1{\cdot}4)$$

$$= (1-t)^{-1-b-k} \sum_{r=0}^{\infty} \frac{(1+b+k)_r}{r!\, (1+b)_r} \; {}_1F_1\left[\begin{matrix} 1+b+k+r\,; \\ 1+b+r\,; \end{matrix} \; \frac{-yt}{1-t}\right] \frac{(-vy)^r}{(1-t)^{2r}}$$

$$\times \left(1-\frac{v}{1-t}\right)^{-1-b-k-r}, \text{ by } (1{\cdot}5)$$

$$= (1-t)^{-1-b-k} \sum_{r,p=0}^{\infty} \frac{(1+b+k)_{r+p}}{r!\, p!\, (1+b)_r} \; {}_1F_1\left[\begin{matrix} 1+b+k+r \\ 1+b+r\,; \end{matrix} \; \frac{-yt}{1-t}\right] \frac{(-y)^r\, v^{r+p}}{(1-t)^{2r+p}}$$

$$= (1-t)^{-1-b-k} \exp\left(\frac{-yt}{1-t}\right) \sum_{r=0}^{\infty} \sum_{p=0}^{r} \frac{(1+b+k)_r}{(r-p)!\, p!\, (1+b)_{r-p}} \; {}_1F_1\left[\begin{matrix} -k\,; \\ 1+b+r-p; \end{matrix} \; \frac{yt}{1-t}\right]$$

$$\times \frac{(-y)^{r-p}\, v^r}{(1-t)^{2r-p}},$$

by (1·6). We can now obtain (3·1) by comparing the coefficients of v^r.

Hence, L. H. S. of (2·1)

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(1+b+k)_n\, (-n)_r\, x^r}{(1+b)_n\, r!\, (1+a)_r} L_n^{(b)}(y)\, t^n$$

$$= \sum_{n,r=0}^{\infty} \frac{(1+b+k)_{n+r}}{(1+b)_{n+r}} \frac{(n+r)!}{n!\, r!} L_{n+r}^{(b)}(y) \frac{(-xt)^r}{(1+a)_r}$$

$$= (1-t)^{-1-b-k} \exp\left(\frac{-yt}{1-t}\right) \sum_{r=0}^{\infty} \sum_{p=0}^{r} \frac{(1+b+k)_r\,(-xt)^r}{(r-p)!\,p!\,(1+b)_{r-p}\,(1+a)_r}\; {}_1F_1\left[\begin{matrix} -k\,; & \dfrac{yt}{1-t} \\ 1+b+r-p\,; & \end{matrix}\right] \times \frac{(-y)^{r-p}}{(1-t)^{2r-p}}, \text{ by } (3\cdot1)$$

$$= (1-t)^{-1-b-k} \exp\left(\frac{-yt}{1-t}\right) \sum_{r,p=0}^{\infty} \frac{(1+b+k)_{r+p}\,(-xt)^{r+p}}{r!\,p!\,(1+b)_r\,(1+a)_{r+p}}\; {}_1F_1\left[\begin{matrix} -k\,; & \dfrac{yt}{1-t} \\ 1+b+r\,; & \end{matrix}\right] \times \frac{(-y)^r}{(1-t)^{2r+p}}$$

$$= (1-t)^{-1-b-k} \exp\left(\frac{-yt}{1-t}\right) \sum_{r=0}^{\infty} \frac{(1+b+k)_r}{r!\,(1+a)_r\,(1+b)_r} \left\{\frac{xyt}{(1-t)^2}\right\}^r {}_1F_1\left[\begin{matrix} 1+b+k+r\,; & \dfrac{-xt}{1-t} \\ 1+a+r\,; & \end{matrix}\right] \times {}_1F_1\left[\begin{matrix} -k\,; & \dfrac{yt}{1-t} \\ 1+b+r\,; & \end{matrix}\right]$$

$$= (1-t)^{-1-b-k} \exp\left[\frac{-t\,(x+y)}{1-t}\right] \sum_{r=0}^{\infty} \frac{(1+b+k)_r}{r!\,(1+a)_r\,(1+b)_r} \left\{\frac{xyt}{(1-t)^2}\right\}^r \times {}_1F_1\left[\begin{matrix} a-b-k\,; & \dfrac{xt}{1-t} \\ 1+a+r\,; & \end{matrix}\right] {}_1F_1\left[\begin{matrix} -k\,; & \dfrac{yt}{1-t} \\ 1+b+r\,; & \end{matrix}\right], \text{ by } (1\cdot6)\,;$$

which proves (2·1)

When k is a postive integer, the formula (2·1) can be written as follows :

(3·2) $$\sum_{n=0}^{\infty} \frac{n!\,\Gamma(1+b+k+n)}{\Gamma(1+a+n)\,\Gamma(1+b+n)}\, L_n^{(a)}(x)\, L_n^{(b)}(y)\, t^n$$

$$= k!\,(1-t)^{-1-b-k} \exp\left[\frac{-t(x+y)}{1-t}\right] \sum_{r=0}^{\infty} \frac{1}{r!\,\Gamma(1+a+r)} \frac{(xyt)^r}{(1-t)^{2r}}\; {}_1F_1\left[\begin{matrix} a-b-k\,; & \dfrac{xt}{1-t} \\ 1+a+r\,; & \end{matrix}\right] \times L_k^{(b+r)}\left(\frac{yt}{1-t}\right).$$

4. *Proof of* (2·2). In (2·2),

$$\text{R. H. S.} = (1-t)^{-1-b} \exp\left(\frac{-yt}{1-t}\right) \sum_{m,k,p=0}^{\infty} \frac{(d)_{m+k+p}\,(a-b)_k}{m!\,k!\,p!\,(1+a)_{k+p}} \left(\frac{-xt}{1-t}\right)^m \left(\frac{xt}{1-t}\right)^k \frac{(xyt)^p}{(1-t)^{2p}}$$

$$= (1-t)^{-1-b} \exp\left(\frac{-yt}{1-t}\right) \sum_{k,p=0}^{\infty} \frac{(d)_{k+p}(a-b)_k}{k!\,p!\,(1+a)_{k+p}} \left(\frac{xt}{1-t}\right)^k \frac{(xyt)^p}{(1-t)^{2p}} \sum_{m=0}^{k} \frac{(-k)_m(-a-k-p)_m}{m!\,(1-a+b-k)_m}$$

$$= (1-t)^{-1-b} \exp\left(\frac{-yt}{1-t}\right) \sum_{k,p=0}^{\infty} \frac{(d)_{k+p}(a-b)_k}{k!\,p!\,(1+a)_{k+p}} \left(\frac{xt}{1-t}\right)^k \frac{(xyt)^p}{(1-t)^{2p}} \frac{(1+b+p)_k}{(1-a+b-k)_k}$$

$$= (1-t)^{-1-b} \exp\left(\frac{-yt}{1-t}\right) \sum_{k,p=0}^{\infty} \frac{(d)_{k+p}\,(1+b)_{k+p}}{k!\,p!\,(1+a)_{k+p}\,(1+b)_p} \left(\frac{-xt}{1-t}\right)^k \frac{(xyt)^p}{(1-t)^{2p}}$$

$$= (1-t)^{-1-b} \exp\left(\frac{-yt}{1-t}\right) \sum_{k=0}^{\infty} \frac{(d)_k}{(1+a)_k} \left(\frac{-xt}{1-t}\right)^k L_k^{(b)}\left(\frac{y}{1-t}\right), \text{ by } (1\cdot1)$$

$$= \sum_{n,k=0}^{\infty} \frac{(d)_k}{(1+a)_k} (-xt)^k \frac{(n+k)!}{n!\,k!} L_{n+k}^{(b)}(y)\, t^n, \text{ by } (1\cdot3)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-n)_k\,(d)_k}{k!\,(1+a)_k} x^k L_n^{(b)}(y)\, t^n = \sum_{n=0}^{\infty} {}_2F_1\left[\begin{matrix} -n, d; \\ 1+a; \end{matrix}\; x\right] L_n^{(b)}(y) t^n,$$

which proves (2·2).

5. *Proof of* (2·3). We first establish the lemma :

$$(5\cdot1) \qquad \sum_{n=0}^{\infty} \frac{(n+k)!}{n!\,k!} L_{n+k}^{(b-n-k)}(y)\, t^n = e^{-yt}(1+t)^{b-k} L_k^{(b-k)}\{y(1-t)\}.$$

Consider the series

$$\sum_{n,k=0}^{\infty} \frac{(n+k)!}{n!\,k!} L_{n+k}^{(b-n-k)}(y)\, t^n v^k = \sum_{n=0}^{\infty} \sum_{k=0}^{n} L_n^{(b-n)}(y) \frac{n!\, t^{n-k} v^k}{(n-k)!\,k!}$$

$$= \sum_{n=0}^{\infty} L_n^{(b-n)}(y)(v+t)^n = (1+v+t)^b e^{-y(v+t)}, \text{ by } (1\cdot2)$$

$$= (1+t)^b e^{-yt} \left(1+\frac{v}{1+t}\right)^b e^{-yv} = (1+t)^b e^{-yt} \sum_{k=0}^{\infty} L_k^{(b-k)}\{y(1+t)\}$$

$$\times \left(\frac{v}{1-t}\right)^k, \text{ by } (1\cdot2)$$

whence we obtain (5·1) by comparing the coefficients of v^k.

Hence, the L. H. S. of (2·3)

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-n)_k\,(d)_k}{k!\,(1+a)_k} x^k L_n^{(b-n)}(y)\, t^n = \sum_{k=0}^{\infty} \frac{(d)_k\,(-xt)^k}{(1+a)_k}$$

$$\times \left[\sum_{n=0}^{\infty} \frac{(n+k)\,!}{n\,!\,k\,!}\, L_{n+k}^{(b-n-k)}(y)\, t^n \right]$$

$$= \sum_{k=0}^{\infty} \frac{(d)_k\,(-xt)^k}{(1+a)_k}\, e^{-yt}\,(1+t)^{b-k}\, L_k^{(b-k)}\{y(1+t)\}, \text{ by } (5\cdot 1)$$

$$= e^{-yt}(1+t)^b \sum_{k=0}^{\infty} \sum_{r=0}^{k} \frac{(d)_k}{(1+)_k}\left(\frac{-xt}{1+t}\right)^k \frac{(1+b-k)_k\,(-k)_r}{k!\,r!\,(1+b-k)_r}\, y^r\,(1+t)^r$$

$$= e^{-yt}(1+t)^b \sum_{k,r=0}^{\infty} \frac{(d)_{k+r}(-b)_k}{k!\,r!\,(1+a)_{k+r}}\left(\frac{xt}{1+t}\right)^k (xyt)^r = e^{-yt}(1+t)^b$$

$$\times\ \Phi_1\left(d,-b\,;\,1+a\,;\,\frac{xt}{1+t},\,xyt\right),$$

which proves (2·3).

6. *Proof of* (2·4). We first prove the lemma :

$$(6\cdot 1) \qquad \sum_{n=0}^{\infty} \frac{(-1)^{n+k}(-b)_{n+k}}{n!\,k\,!}\; {}_2F_1\left[\begin{matrix} -n-k,\ d\ ; \\ 1+b-n-k\ ; \end{matrix}\ y\right] t^n$$

$$= (1+t)^b \sum_{r=0}^{k} \frac{(d)_r\ (-b)_{k-r}}{r\,!\,(k-r)\,!}(-1)^k\,(1+t)^{-k+r}\,(1+yt)^{-d-r}\,y^r.$$

Consider the series

$$\sum_{n,k=0}^{\infty} \frac{(-1)^{n+k}\,(-b)_{n+k}}{(n+k)\,!}\,\frac{(n+k)\,!}{n!\,k!}\; {}_2F_1\left[\begin{matrix} -n-k,\ d\ ; \\ 1+b-n-k\ ; \end{matrix}\ y\right] t^n\,v^k$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^n\ (-b)_n}{n!}\; {}_2F_1\left[\begin{matrix} -n,\ d\ ; \\ 1+b-n\ ; \end{matrix}\ y\right] \frac{n\,!}{(n-k)\,!\,k\,!}\, t^{n-k}\,v^k$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\,(-b)_n}{n\,!}\; {}_2F_1\left[\begin{matrix} -n,\ d\ ; \\ 1+b-n\ ; \end{matrix}\ y\right](t+v)^n = \sum_{n=0}^{\infty} \sum_{r=0} \frac{(-1)^n\ (-b)_n\ (-n)_r\ (d)_r}{n\,!\,r\,!\,(1+b-n)_k}$$

$$\times\ y^r\,(t+v)^n$$

$$= \sum_{n,r=0}^{\infty} \frac{(-1)^n\,(-b)_{n+r}\,(d)_r}{n\,!\,r\,!\,(1+b-n-r)_r}\,y^r\,(t+v)^{n+r} = \sum_{n,r=0}^{\infty} \frac{(-1)^{n+r}\ (-b)_n\ (d)_r}{n\,!\,r\,!}\, y^r\,(t+v)^{n+r}$$

$$= (1+t+v)^b\,[1+y(t+v)]^{-d} = (1+t)^b\left(1+\frac{v}{1+t}\right)^b (1+yt)^{-d}\left(1+\frac{vy}{1+yt}\right)^{-d}$$

$$= (1+t)^b (1+yt)^{-d} \sum_{k=0}^{\infty} \frac{(-b)_k}{k\,!} \left(\frac{-v}{1+t}\right)^k \sum_{r=0}^{\infty} \frac{(d)_r}{r\,!} \left(\frac{-vy}{1+yt}\right)^r$$

$$= (1+t)^b (1+yt)^{-d} \sum_{k=0}^{\infty} \sum_{r=0}^{k} \frac{(d)_r\,(-b)_{k-r}}{r\,!\,(k-r)\,!} (-v)^k (1+t)^{-k+r} (1+yt)^{-r} y^r,$$

whence (6·1) follows on comparing the coefficients of v^k.

Hence, the R. H. S. of (2·4) $= (1+t)^b \sum_{p,k,r=0}^{\infty} \frac{(d)_{p+r}\,(b)_k}{p\,!\,k\,!\,r\,!\,(1+a)_{k+r}} (-yt)^p$

$$\times \left(\frac{xt}{1+t}\right)^k (xyt)^r$$

$$= (1+t)^b \sum_{k,r=0}^{\infty} \left[\frac{(d)_r\,(-b)_k}{k\,!\,r\,!\,(1+a)_{k+r}} \left(\frac{xt}{1+t}\right)^k (xyt)^r \sum_{p=0}^{\infty} \frac{(d+r)_p}{p\,!} (-yt)^p\right]$$

$$= (1+t)^b \sum_{k,r=0}^{\infty} \frac{(d)_r\,(-b)_k}{k\,!\,r\,!\,(1+a)_{k+r}} \left(\frac{xt}{1+t}\right)^k (xyt)^r (1+yt)^{-d-r}$$

$$= (1+t)^b \sum_{k=0}^{\infty} \sum_{r=0}^{k} \frac{(d)_r\,(-b)_{k-r}}{r\,!\,(k-r)\,!\,(1+a)_k} (xt)^k (1+t)^{-k+r} (1+yt)^{-d-r} y^r$$

$$= \sum_{n,k=0}^{\infty} \frac{(-1)^{n+k}\,(-b)_{n+k}}{n\,!\,k\,!\,(1+a)_k} (-xt)^k \; {}_2F_1\left[\begin{matrix}-n-k,\ d\,;\\ 1+b-n-k\,;\end{matrix}\ y\right] t^n, \text{ by (6·1)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\,(-b)_n}{(1+a)_n} L_n^{(a)}(x) \; {}_2F_1\left[\begin{matrix}-n,\ d\,;\\ 1+b-n\,;\end{matrix}\ y\right] t^n; \text{ which proves (2·4)}$$

7. *Particular cases*

Putting $k = 0$ in (2·1), we have

$$\text{(7·1)} \qquad \sum_{n=0}^{\infty} \frac{n\,!}{(1+a)_n} L_n^{(a)}(x)\, L_n^{(b)}(y)\, t^n$$

$$= (1-t)^{-1-b} \exp\left[\frac{-t(x+y)}{1-t}\right] \Phi_3\left[a-b\,;\, 1+a\,;\ \frac{xt}{1-t},\ \frac{xyt}{(1-t)^2}\right],$$

where $\Phi_3(b\,;\,c\,;\,x,y) \equiv \sum_{m,n=0}^{\infty} \frac{(b)_m\, x^m\, y^n}{m\,!\,n\,!\,(c)_{m+n}}$ is Humbert's confluent hypergeometric function of two variables [3 (vol. 1); p. 225]. This is formula (1) in[6].

Putting $b = a$ in (3·2), we get

$$\text{(7·2)} \qquad \sum_{n=0}^{\infty} \frac{n\,!\,\Gamma(1+a+k+n)}{\{\Gamma(1+a+n)\}^2} L_n^{(a)}(x)\, L_n^{(a)}(y)\, t^n$$

$$= k\,!\,(1-t)^{-1-a-k}\exp\left[\frac{-t(x+y)}{1-t}\right]\sum_{r=0}^{\infty}\left[\frac{L_k^{(a+r)}\left(\frac{xt}{1-t}\right)}{r\,!\,\Gamma(1+a+k+r)}\,L_k^{(a+r)}\left(\frac{yt}{1-t}\right)\left\{\frac{xyt}{(1-t)^2}\right\}^{r}\right]$$

Bottema[1] has given a much more complicated expression for the summation on the left of (7·2).

Taking $k = 0$ and $a = b$ in (2·1), we obtain

$$(7\cdot3)\qquad \sum_{n=0}^{\infty}\frac{n\,!}{(1+a)_n}L_n^{(a)}(x)\,L_n^{(a)}(y)\,t^n = (1-t)^{-1-a}\exp\left[\frac{-t(x+y)}{1-t}\right]{}_1F_0\left[\begin{matrix}--\,; & \frac{xyt}{(1-t)^2}\\ 1+a\,; & \end{matrix}\right].$$

This is Hardy-Hille formula [3 (vol. 2) ; p. 190].

Taking $b = a$ in (2·2), we have

$$(7\cdot4)\qquad \sum_{n=0}^{\infty}{}_2F_1(-n,\,d\,;\,1+a\,;\,x)\,L_n^{(a)}(y)\,t^n$$

$$= (1-t)^{-1-a}\exp\left(\frac{-yt}{1-t}\right)(1-t+xt)^{-d}\,{}_1F_1\left[\begin{matrix}d\,; & \frac{xyt}{(1-t)(1-t+xt)}\\ 1+a\,; & \end{matrix}\right],$$

which is due to Brafman[2].

Changing x to x/d in (2·3) and then making $d \to \infty$, we obtain

$$(7\cdot5)\qquad \sum_{n=0}^{\infty}\frac{n\,!}{(1+a)_n}L_n^{(a)}(x)\,L_n^{(b-n)}(y)\,t^n = e^{-yt}(1+t)^b\,\Phi_3\left(-b\,;\,1+a\,;\,\frac{xt}{1+t},\,xyt\right).$$

This is formula (16) in[6] and can also be obtained from (2·4) if we change y to y/d and then let $d \to \infty$.

Finally, we may mention that (7·1) can also be derived from (2·3) by changing x to x/d and making $d \to \infty$.

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REFERENCES

1. Bottema, O. On a generalisation of the formula of Hille and Hardy in the theory of Laguerre Polynomials. *Proc Kon. Nad. Acad v. Wet.*, 49 : 1032–1036, (1946).
2. Brafman, F. Some generating functions for Laguerre and Hermite polynomials. *Canad. J Math.* 5 : 301–305, (1953).
3. Erdelyi, A. *et al*, Higher Transcendental Functions. *New York*, (1953).
4. Jain, R. N. Confluent Hypergeometric Functions of Three Variables. *Proc. Nat Acad. Sci. India*, (To appear).
5. Rainville, E. D. Special Functions, *New York*, (1960).
6. Rangarajan, S. K. Series involving products of Laguerre polynomials. *Proc. Ind. Acad Sci Sec. A.* 58 : 362–367, (1963).
7. Slater, L. J. The Confluent Hypergeometric Functions, *Cambridge*, (1960).

ON CERTAIN INTEGRALS INVOLVING WHITTAKER'S FUNCTION

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1. INTRODUCTION

The integral equation

$$\psi(p) = p\int_0^\infty e^{-px} f(x)\, dx,\ R(p) > 0 \tag{1·1}$$

is called the Laplace transform and is symbollically denoted by

$$\psi(p) \doteqdot f(x) \text{ or } f(x) \doteqdot \psi\ (p) \tag{1·2}$$

Meijer [6, p. 727] introduced the generalised Laplace transform

$$F(p) = \int_0^\infty e^{-\frac{1}{2}px}\, W_{k+\frac{1}{2},m}\,(px)\,(px)^{-k-\frac{1}{2}} f(x)\, dx,\ R(p) > 0, \tag{1·3}$$

where $W_{k,m}(x)$ is the confluent hypergeometric function. The integral equation in (1·3) can be denoted as

$$\varphi(p) \overset{k+\frac{1}{2}}{\underset{m}{\leftarrow}} f(x), \tag{1·4}$$

where $\varphi(p) = p\,F(p)$. We shall call this relation the Meijer transform of $f(x)$. If we substitute $k = m$ in (1·4), then due to the identity

$$e^{-\frac{1}{2}pt} = (pt)^{-m-\frac{1}{2}}\, W_{m+\frac{1}{2},m}\,(pt),$$

it reduces to (1·1), where $\psi(p) = \varphi(p)$.

The aim of this paper is to evaluate certain integrals of the form

$$\int_0^\infty e^{\frac{1}{2}px}\, W_{k,\mu}\,(px)\, f(x)\, dx.$$

We have given two theorems and with the help of these, we have evaluated certain integrals which we believe to be new.

2. *Theorem* 1. Let (*i*) $p\ \varphi(p) \doteqdot h(t)\ K(t),$ (2·1)

(*ii*) $p\ K(p) \doteqdot t^{\mu}\ \psi(t),$ (2·2)

(*iii*) $p\ \chi(p) \doteqdot h(t),$ (2·3)

and (*iv*) $p\ \psi(p) \overset{k+\frac{1}{2}}{\underset{m}{\leftarrow}} g(t).$ (2·4)

Then

$$\varphi(p) = \int_0^\infty g(y)\, dy \int_0^\infty e^{-\frac{1}{2}yx}\, W_{k+\frac{1}{2},m}\,(yx)\,(yx)^{-k-\frac{1}{2}}\, x^{\mu}\, \chi(p+x)\, dx, \tag{2·5}$$

provided the various changes in the order of integrations are permissible and the double integral on the right exists.

Proof. We have [from (2·1)]

$$\varphi(p) = \int_0^\infty e^{-pt}\, h(t)\, K(t)\, dt$$

$$= \int_0^\infty e^{-pt}\, h(t)\, dt \int_0^\infty e^{-tx}\, x^{\mu}\, \psi(x)\, dx, \text{ [using (2·2)]}$$

$$= \int_0^\infty x^{\mu}\, \psi(x)\, dx \int_0^\infty e^{-(p+x)t}\, h(t)\, dt$$

$$= \int_0^\infty x^{\mu}\, \psi(x)\, \chi(p+x)\, dx; \text{ [using (2·3)]}$$

$$= \int_0^\infty x^{\mu}\, \chi(p+x)\, dx \int_0^\infty e^{-\frac{1}{2}yx}\, W_{k+\frac{1}{2},m}(yx)\, (yx)^{-k-\frac{1}{2}}\, g(y)\, dy, \text{ [using (2·4)]}$$

$$= \int_0^\infty g(y)\, dy \int_0^\infty e^{-\frac{1}{2}yx}\, W_{k+\frac{1}{2},m}(yx)\, (yx)^{-k-\frac{1}{2}}\, x^{\mu}\, \chi(p+x)\, dx.$$

Hence the theorem.

Corollery. Let us put $k = m$, we get that

if (*i*) $p\,\varphi(p) \doteqdot h(t)\, K(t)$, (*ii*) $p\, K(p) \doteqdot t^{\mu}\, \psi(t)$,

(*iii*) $p\, \chi(p) \doteqdot h(t)$ and (*iv*) $p\, \psi(p) \doteqdot g(t)$.

Then under the conditions of the theorem

$$\varphi(p) = \int_0^\infty g(y)\, dy) \int_0^\infty e^{-yx}\, x^{\mu}\, \chi(p+x)\, dx. \tag{2·6}$$

Theorem 2. Let (*i*) $p\,\varphi(p) \doteqdot h(t)\, K(t)$,

(*ii*) $p\, K(p) \doteqdot t^{\mu}\, \psi(1/t)$,

(*iii*) $p\, \chi(p) \doteqdot h(t)$

and (*iv*) $p\, \psi(p) \overset{k+\frac{1}{2}}{\underset{m}{\leftarrow}} g(t)$.

Then

$$\varphi\left(\frac{1}{p}\right) = \int_0^\infty g(y)\, dy \int_0^\infty e^{-\frac{1}{2}yx}\, W_{k+\frac{1}{2},m}(yx)\, (yx)^{-k-\frac{1}{2}}\, \chi\left(\frac{p+x}{px}\right) x^{-\mu-2}\, dx, \tag{2·7}$$

provided the various changes in the order of integrations are permissible and the double integral on the right exists.

The proof is on the same lines as in theorem 1.

Corollery. Let us put $k = m$, we get that

if (*i*) $p\,\varphi(p) \doteqdot h(t)\, K(t)$, (*ii*) $p\, K(p) \doteqdot t^{\mu}\, \psi(1/t)$,

(*iii*) $p\, \chi(p) \doteqdot h(t)$ and (*iv*) $p\, \psi(p) \doteqdot g(t)$.

Then under the conditions of the theorem

$$\varphi(1/p) = \int_0^\infty g(y)\, dy \int_0^\infty e^{-yx}\, \chi\left(\frac{p+x}{px}\right) x^{-\mu-2}\, dx. \tag{2·8}$$

3. Example on theorem 1.

$$\text{Let } \chi(p) = p^{2k-\mu-1},\ h(t) = \frac{t^{\mu-2k}}{\Gamma(1+\mu-2k)}$$

Further let $\psi(t) = t^{-1-v} e^{-1/t}$, $K(p) = 2p^{\frac{1}{2}v-\frac{1}{2}\mu} K_{\mu-v}(2\sqrt{p})$.

$$\varphi(p) = \Gamma(1+v-2k)\, p^{2k-\frac{1}{2}\mu-\frac{1}{2}v-\frac{1}{2}} e^{1/2p} W_{2k-\frac{1}{2}\mu-\frac{1}{2}v-\frac{1}{2},\frac{1}{2}\mu-\frac{1}{2}v}(1/p).$$

$$g(t) = \frac{\Gamma(1+v-2k)}{\Gamma(1+v-k\pm m)} t^v \, {}_1F_2\left[\begin{matrix}1+v-2k\,; \\ 1+v-k\pm m\,;\end{matrix}\ -t\right] \text{[4, p. 123]}$$

Substituting the values of $\chi(p)$, $\varphi(p)$ and $g(y)$ in (2·5) and evaluating the x–integral [3, p. 237], we get

$$\int_0^\infty e^{\frac{1}{2}py} W_{k-\mu-\frac{1}{2},m}(py)\, y^{v-k-\frac{1}{2}}\, {}_1F_2\left[\begin{matrix}1+v-2k\,; \\ 1+v-k\pm m\,;\end{matrix}\ -y\right] dy$$

$$= \frac{\Gamma(1+\mu-2k)\,\Gamma(1+v-k\pm m)}{\Gamma(1+\mu-k\pm m)} p^{k-\frac{1}{2}\mu-\frac{1}{2}v} e^{1/2p} W_{2k-\frac{1}{2}\mu-\frac{1}{2}v-\frac{1}{2},\frac{1}{2}\mu-\frac{1}{2}v}(1/p), \tag{3·1}$$

$$R(1+\mu-2k) > 0,\ R(\tfrac{1}{2}+2\mu-v) > 0 \text{ and } R(1+v-k\pm m) > 0.$$

Example on theorem 2.

Let $$\psi(p) = p^\mu \tan^{-1}(1/p),$$

$$g(t) = \frac{\Gamma(1-\mu-2k)t^{-\mu}}{\Gamma(1-\mu-k\pm m)}\, {}_4F_5\left[\begin{matrix}\frac{1}{2}, 1, \frac{1}{2}-\frac{1}{2}\mu-k, 1-\frac{1}{2}\mu-k\,; \\ \frac{3}{2}, \frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}k\pm\frac{1}{2}m, 1-\frac{1}{2}\mu-\frac{1}{2}k\pm\frac{1}{2}m\,;\end{matrix}\ -\frac{t^2}{4}\right] \text{[7, p. 71]}$$

$$K(p) = -(1/p)\,[ci(p)\sin(p) + si(p)\cos(p)].$$

Further let $$h(t) = t,\ \therefore\ \chi(p) = p^{-2}.$$

$$\varphi(p) = -\int_0^\infty e^{-pt}\,[ci(t)\sin(t) + si(t)\cos(t)]\,dt.$$

Now $$ci(t) = -Ci(t) \text{ and } si(t) = Si(t) - \pi/2,$$

$$\therefore\ \varphi(p) = \int_0^\infty e^{-pt}\left\{Ci(t)\sin(t) - \left[Si(t) - \frac{\pi}{2}\right]\cos(t)\right\} dt$$

$$= \int_0^\infty e^{-pt}\{Ci(t)\sin(t) - Si(t)\cos(t)\}dt + \frac{\pi}{2}\int_0^\infty e^{-pt}\cos(t)\,dt.$$

$$= (1+p^2)^{-1}\,[\tfrac{1}{2}\pi p - \log(p)] \quad \text{[2, p. 178 and 154]}.$$

Substituting the values of $\chi(p)$, $\varphi(p)$ and $g(t)$ in (2·7) and evaluating the x–integral [3, p. 237] after putting $k = -\frac{1}{2} - \frac{1}{2}\mu$, we get

$$\int_0^\infty e^{\frac{1}{2}py} W_{\frac{1}{2}-\mu-1,m}(py)\, {}_3F_4\left[\begin{matrix}\frac{1}{2}, 1, 1\,; \\ \frac{3}{4}-\frac{\mu}{4}\pm\frac{1}{2}m, \frac{5}{4}-\frac{\mu}{4}\pm\frac{1}{2}m\,;\end{matrix}\ -\frac{y^2}{4}\right] y^{-\frac{1}{2}\mu}\, dy$$

$$= p^{1+\frac{1}{2}\mu}(1+p^2)^{-1}\left[\frac{\pi}{2p} + \log(p)\right], \tag{3·2}$$

$$R(\tfrac{3}{2}-\tfrac{1}{2}\mu\pm m) > 0,\ R(\mu) < 1.$$

We shall now give a number of integrals calculated from these theorems and their corollaries in the form of a table.

1. $$\int_0^\infty e^{\frac{1}{2}py} W_{k-\mu-\frac{1}{2},m}(py)\, y^{v-k-\lambda-\frac{1}{2}}\, {}_2F_2\left[\begin{matrix} v, 1+v-\lambda-2k\,; \\ 1+v-\lambda-k\pm m\,; \end{matrix} -y\right] dy$$

$$= \frac{\Gamma(\mu+\lambda)\,\Gamma(1+\mu-2k)\,\Gamma(1+v-\lambda-k\pm m)}{\Gamma(1+v+\mu-2k)\,\Gamma(1+\mu-k\pm m)}\, p^{\frac{1}{2}-k}\, {}_2F_1\left[\begin{matrix} 1+\mu-2k, 1+v-\lambda-2k\,; \\ 1+v+\mu-2k\,; \end{matrix} 1-p\right],$$

$R(\mu+\lambda) > 0$, $R(1+\mu-2k) > 0$ and $R(1+v-\lambda-k\pm m) > 0$.

2. $$\int_0^\infty e^{\frac{1}{2}py} W_{-k-\frac{1}{2},m}(py)\, y^{k-\frac{1}{2}}\, {}_3F_4\left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, 1\,; \\ \frac{1}{2}+\frac{1}{2}k\pm\frac{1}{2}m,\ 1+\frac{1}{2}k\pm\frac{1}{2}m\,; \end{matrix} -\frac{y^2}{4}\right] dy$$

$$= \frac{p^{\frac{1}{2}-k}}{\sqrt{p^2+1}}\left[\log\left(\frac{\sqrt{p^2+1}+1}{p}\right) + \sinh^{-1}(p)\right],$$

$R(k) > -1/4$, and $R(1+k\pm m) > 0$.

3. $$\int_0^\infty e^{\frac{1}{2}py} W_{k-\mu-\frac{1}{2},m}(py)\, y^{\mu+n-k-1}\, {}_2F_3\left[\begin{matrix} \frac{1}{2}+n, \frac{1}{2}+\mu+n-2k\,; \\ \frac{1}{2}, \frac{1}{2}+\mu+n-k\pm m\,; \end{matrix} -\frac{1}{2}y\right] dy$$

$$= \frac{(-1)^n\,\pi\,\Gamma(1+2\mu+2n-4k)\,\Gamma(\frac{1}{2}+\mu+n-k\pm m)\, p^{k-\mu-n}}{2^{\mu+n-2k-\frac{1}{2}}\,\Gamma(\frac{1}{2}+n)\,\Gamma(\frac{1}{2}+\mu+n-2k)\,\Gamma(1+\mu-k\pm m)}\, e^{\frac{1}{4p}}\, D_{4k-2\mu-2n-1}\left(\frac{1}{\sqrt{p}}\right);$$

n is a positive integer, $R(\frac{1}{2}\mu-n+\frac{1}{2}) > 0$, $R(1+\mu-2k) > 0$ and $R(\frac{1}{2}+\mu+n-k\pm m) > 0$.

4. $$\int_0^\infty e^{-y(a-p)}\, \mathrm{Erfc}(\sqrt{py})\, \frac{dy}{\sqrt{y}} = \frac{2}{\sqrt{\pi}}(p-a)^{-\frac{1}{2}} \log\left(\frac{\sqrt{p-a}+\sqrt{p}}{\sqrt{a}}\right),\ p > a > 0.$$

5. $$\int_0^\infty e^{\frac{1}{2}py} W_{-\frac{1}{4}-\frac{1}{2}a-\frac{1}{2}\lambda,\,-\frac{1}{4}-\frac{1}{2}\lambda+\frac{1}{2}a}(py)\, y^{\frac{1}{2}\lambda+\frac{1}{2}a-\frac{3}{4}}\, e^{-y}\, L_n^{a}(y)\, dy$$

$$= \frac{\sqrt{\pi}\,2^{a+n-\lambda}\,\Gamma(1+a+n)\,\Gamma(\frac{1}{2}+a+n)\,\Gamma(2+2\lambda)\, p^{\frac{1}{2}n+\frac{3}{4}}}{\Gamma(1+n)\,\Gamma(1+\lambda)\,\Gamma(\frac{1}{2}+a)\,(p-1)^{\frac{1}{2}a+\frac{1}{2}n+\frac{1}{2}\lambda+\frac{1}{2}}}\, P_{\lambda-a-n}^{-a-n-\lambda-1}(p^{-\frac{1}{2}}),$$

$R(a) > -1$ and $R(\lambda) > -\frac{3}{2}$.

6. $$\int_0^\infty e^{\frac{1}{2}py} W_{-\frac{1}{2}\lambda-\frac{1}{2},\frac{1}{2}}(py)\, y^{\frac{1}{2}\lambda-1}\left[\sin(v\pi)\, J_{2v}(\sqrt{8ay}) + \cos(v\pi)\, Y_{2v}(\sqrt{8ay})\right] dy$$

$$= -\frac{\Gamma(\frac{1}{2}+\lambda\pm v)\, p^{\frac{1}{2}-\frac{1}{2}\lambda}}{\sqrt{2a}\,\Gamma(1+\lambda)}\, e^{a/p}\, W_{-\lambda,v}\left(\frac{2a}{p}\right),\ -\frac{1}{2} < R(v) < \frac{1}{2} \text{ and } R(\lambda+\frac{1}{2}) > |v|.$$

7. $$\int_0^\infty e^{\frac{1}{2}py} W_{-\frac{1}{2}\lambda-\frac{1}{2},\frac{1}{2}\lambda}(py)\, \sin(2\sqrt{ay})\, y^{\frac{1}{2}\lambda-\frac{3}{2}}\, dy$$

$$= \frac{\pi}{\lambda}\, p^{\frac{1}{2}-\frac{1}{2}\lambda} - \frac{\pi\,\Gamma(2\lambda)\, p^{\frac{1}{2}-\frac{1}{2}\lambda}}{2^{\lambda-1}\,\Gamma(1+\lambda)}\, e^{a/2p}\, D_{-2\lambda}\left(\sqrt{\frac{2a}{p}}\right),\ R(\lambda) > -\frac{1}{2}.$$

8. $$\int_0^\infty e^{\frac{1}{2}py}\, W_{-\frac{1}{2}\mu-\frac{1}{2},\frac{1}{2}\mu}(py)\, e^{-a/2y}\, W_{k,v}\left(\frac{a}{y}\right) y^{-k-\frac{1}{2}\mu-\frac{1}{2}}\, dy$$
$$= \frac{2^{1+2\mu+2k}\sqrt{a}\,\Gamma(\frac{1}{2}+\mu+k\pm v)}{\Gamma(1+\mu)}\, p^{\frac{1}{2}\mu+k}\, S_{-2\mu-2k,2v}\,(2\sqrt{ap}),$$
$R(\frac{1}{2}+\mu+k\pm v)>0$, $\frac{1}{2}-\mu-k\pm v\neq 0$ and $-\frac{1}{2}<R(v)<\frac{1}{2}$.

9. $$\int_0^\infty e^{\frac{1}{2}py}\, W_{-3/2,0}(py)\cdot\{[\log(ry)]^2-\tfrac{1}{6}\pi^2\}\frac{dy}{\sqrt{y}} = p^{-\frac{1}{2}}\,[\log(p)]^2+\tfrac{1}{3}\pi^2 p^{-\frac{1}{2}}.$$

10. $$\int_0^\infty e^{\frac{1}{2}py}\, W_{-\frac{3v}{2},\frac{v}{2}}(py)\,[J_v(\sqrt{2ay})]^2\, y^{\frac{1}{2}v-\frac{1}{2}}\, dy$$
$$= \frac{2^{2v-\frac{1}{2}}\, a^{v-\frac{1}{2}}\,\Gamma(v)\, p^{-\frac{3v}{2}}}{\sqrt{\pi}\,\Gamma(\frac{1}{2}+2v)}\, e^{\frac{a}{p}}\, D_{-2v}\,(2\sqrt{a/p}),\ R(v)>0.$$

11. $$\int_0^\infty e^{\frac{1}{2}py}\, W_{-\frac{3}{4}-\lambda+\frac{1}{2}v,\frac{1}{4}-\frac{1}{2}v}(py)\, y^{\frac{1}{2}v-\frac{5}{4}}\, e^{-\frac{1}{2}y}\,[D_{2v}(\sqrt{2y})+D_{2v}(-\sqrt{2y})]\, dy$$
$$= \frac{\pi^{3/2}\, 2^{2v-\lambda+\frac{1}{2}}\,\Gamma(v)\,\Gamma(2+2\lambda)\, p^{\frac{3}{4}+\frac{1}{2}v}}{\Gamma(\frac{1}{2}-v)\,\Gamma(1+\lambda)\,\Gamma(\frac{3}{2}+\lambda-v)\,(1-p)^{\frac{3}{4}+\frac{1}{2}v+\frac{1}{2}\lambda}}\, P^{-\frac{1}{2}-v-\lambda}_{\frac{1}{2}-v+\lambda}(\sqrt{p}),\ R(v)>0,\ R(\lambda)>-1.$$

12. $$\int_0^\infty e^{\frac{1}{2}py}\, W_{\frac{3}{4}v-\lambda,\frac{3}{4}v+\frac{1}{2}}(py)\, e^{-\frac{1}{4}y}\,[I_{\frac{1}{2}v-\frac{1}{2}}(\tfrac{1}{4}y)-I_{\frac{1}{2}v+\frac{1}{2}}(\tfrac{1}{4}y)]\, y^{-\frac{1}{2}-\frac{1}{4}v}\, dy$$
$$= \frac{\Gamma(1+\frac{1}{2}v)\,\Gamma(-\frac{1}{2}v)\,\Gamma(1+\lambda-\frac{1}{2}v)\, p^{-\frac{3}{4}v}}{\sqrt{\pi}\, 2^{\frac{1}{2}v-\frac{3}{2}}\,\Gamma(1+\lambda)\,(1-2p)^{\frac{1}{2}\lambda-\frac{1}{2}v}}\, P^{v-\lambda}_{\frac{1}{2}v}\left(\frac{1}{p}-1\right),\ R(\lambda-\tfrac{1}{2}v)>0 \text{ and } -1<R(v)<0.$$

13. $$\int_0^\infty e^{\frac{1}{2}py}\, W_{\frac{3}{2}v-\lambda-\frac{1}{4},\frac{3}{2}v+\frac{1}{4}}(py)\, e^{-\frac{1}{4}y}\, I_v(\tfrac{1}{4}y)\, y^{-\frac{3}{4}-\frac{1}{2}v}\, dy$$
$$= \frac{\sin[(\lambda-2v-\frac{1}{2})\pi]\,\Gamma(\frac{1}{2}\pm v)\, p^{\frac{1}{4}-\frac{1}{2}v}}{\pi\, 2^{v-\frac{1}{2}}\sin[(\lambda-v-\frac{1}{2})\pi]\,\Gamma(1+\lambda)\,(1-2p)^{\frac{1}{2}\lambda-v+\frac{1}{4}}}\, Q^{v}_{\lambda-2v-\frac{1}{2}}\left(\frac{1-p}{\sqrt{1-2p}}\right),$$
$0<p<1$, $-\frac{1}{2}<R(v)<\frac{1}{2}$ and $R(\lambda-v+\frac{1}{2})>0$.

14. $$\int_0^\infty e^{\frac{1}{2}py}\, W_{\frac{1}{2}\mu,\frac{1}{2}+\frac{1}{2}\mu}(py)\, y^{\frac{1}{2}\mu-\frac{1}{2}}\,[1+i\sqrt{\pi y}\, e^{y}\,\mathrm{Erf}(i\sqrt{y})]\, dy$$
$$= \frac{\sqrt{\pi}\,\Gamma(\frac{3}{2}+\mu)\,\Gamma(-\frac{1}{2}-\mu)}{\Gamma(-\mu)}\, p^{\frac{1}{2}-\frac{1}{2}\mu}\,(1-p)^{-1}\,(p^{\frac{1}{2}+\mu}-1),\ -\tfrac{3}{2}<R(\mu)<-\tfrac{1}{2}.$$

15. $$\int_0^\infty e^{\frac{1}{2}py}\, W_{v-\frac{3}{4},\frac{1}{4}}(py)\, e^{-\frac{1}{2}y}\,[D_{2v}(-\sqrt{2y})-D_{2v}(\sqrt{2y})]\, y^{-\frac{3}{4}}\, dy$$
$$= \frac{\pi\, 2^{v+1}\, p^{\frac{1}{2}}}{\Gamma(1-v)\,\Gamma(\frac{3}{2}-v)}(1-p)^{-1}\,(p^{v}-1),\ R(v)<1.$$

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REFERENCES

1. Carslaw, H. S Fourier's Series and Integrals.
2. Erdelyi, A. and others Tables of Integral Transforms. Vol I. (1954).
3. Erdelyi, A. and others. Tables of Integral Transforms. Vol 2 (1954)
4. Jain, M. K. On Meijer Transform, *Acta Mathmetica*. Vol. 93 : (1955).
5. Jaiswal, J.P On Meijer Transform. *Mathmatische Zeitshrift*, Band 55, Heft 3 385-98 (1952)
6. Meijer, S. C. Eine neue Erweiterung der Laplace transformation. *Proc Ned. Acad. V. wetensch, Amsterdam*, 44 : (1941 a).

ON THE RADIAL PULSATIONS OF AN INFINITE CYLINDER

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ABSTRACT

The stability of Radial pulsations of a finitely long gravitating cylinder, consisting of a compressible and inviscid fluid has been investigated. The density of the fluid is assumed to vary inversely as the distance from the axis of the cylinder. It is established that the pulsations are stable.

1. INTRODUCTION

The effect of magnetic fields on the gravitational stability of an infinitely long cylinder constituted by a compressible and inviscid but infinitely conducting fluid has recently been studied by various authors (Chandrasekhar and Fermi,[1] Ejnar Lyttkens,[2] Bhatnagar, P. L.[3]).

This paper studies the same problem with the fluid density varying inversely as the distance from the axis of the Cylinder, but the magnetic field is absent.

2. BASIC EQUATIONS

Consider the radial pulsations of an infinitely long Cylinder of self-gravitating fluid. It is assumed that the pulsations are so small that square of amplitudes of displacement and their derivatives can be ignored in comparision to their first powers.

In setting up the Eulerian equations of motion, we shall assume that the motion of individual gas particle takes place adiabatically.

Let the pressure p, density ρ and gravity g in the equilibrium position, at a point distant r, from the axis of the cylinder be altered by δp, $\delta\rho$ and δg, respectively and let the displacement of material from its equilibrium position be δr, then the equations of small oscillations are :

(*i*) the equation of continuity

$$\delta\rho = -\frac{1}{r}\frac{\partial}{\partial r}(\rho r\,\delta r) \tag{1}$$

(*ii*) the equation of motion

$$\rho\sigma^2\delta r = \frac{\partial}{\partial r}\,\delta p + g\delta\rho \tag{2}$$

(*iii*) adiabatic relation

$$\delta p = \rho g\,\delta r - \frac{\gamma p}{r}\frac{\partial}{\partial r}(r\delta r) \tag{3}$$

(*iv*) equation of equilibrium

$$\frac{\partial p}{\delta r} = -\rho g \tag{4}$$

(*v*) where $$g = \frac{2Gm}{r} \tag{5}$$

and γ is the ratio of specific heats, G is the gravitational constant, m is the mass per unit length interior to r.

Let all the physical variables vary as $e^{i\sigma t}$.

Eliminating δp and $\delta\rho$ from the equation (2) with the help of the equations (1) and (3), and putting

$$r\delta r = r\xi, \; r = xR \tag{6}$$

where R is the radius of the cylinder, we get

$$\frac{x^2 d^2\xi}{dx^2} + \frac{x d\xi}{dx}\left[1 - \frac{RG\rho x}{p}\right] + \left[\frac{\sigma^2 R^2 x^2 \rho}{\gamma p} + \frac{2gRx\rho}{\gamma p} - \frac{4\pi G\rho^2 R^2 x^2}{\gamma p} - \frac{\rho g R x}{p} - 1\right]\xi = 0 \tag{7}$$

or,

$$\frac{d^2\xi}{dx^2} + \frac{Q(x)}{x}\frac{d\xi}{dx} + \frac{P(x)}{x^2}\xi = 0, \tag{8}$$

where

$$Q(x) = 1 - \frac{\rho R g x}{p}, \tag{9}$$

$$P(x) = \frac{\sigma^2 R^2 x^2 \rho}{\gamma p} + \frac{2\rho g R x}{\gamma p} - \frac{4\pi G\rho^2 R^2 x^2}{\gamma p} - \frac{\rho g R x}{p} - 1 \tag{10}$$

Now ρ is assumed to be

$$\rho = \frac{\mu}{r} = \frac{\mu}{Rx}, \quad \mu \text{ is a constant.} \tag{11}$$

Then we have

$$g = \frac{2Gm}{r} = 2G\int_0^r \frac{2\pi r \frac{\mu}{r}\, dr}{r} = 4\pi G\mu \quad \ldots \tag{12}$$

$$p = \int_r^R \rho g dr = -4\pi G\mu^2 \log x \tag{13}$$

Substituting these values in the equations (9) and (10), we get

$$P(x) = \left[1 - \frac{1}{\gamma} - \frac{\sigma^2 R x}{4\pi G\mu\gamma}\right]\frac{1}{\log x} - 1 \tag{14}$$

$$Q(x) = 1 + \frac{1}{\log x} \tag{15}$$

Now we change the independent variable from x to y by making the substitution $x = e^y$ such that

$$\frac{dy}{dx} = \frac{1}{e^y} \tag{16}$$

With the help of the equations (14), (15) and (16) the equation (8) transforms to

$$\frac{d^2\xi}{dy^2} + \frac{1}{y}\frac{d\xi}{dy} + \frac{1}{y}\left[A - Be^y - y\right] = 0 \tag{17}$$

where

$$A = 1 - \frac{1}{\gamma}, \tag{18}$$

$$B = \frac{\sigma^2 R}{4\pi G\mu\gamma} \tag{19}$$

we attempt a solution of the equation (8) under the boundary conditions

$$\left.\begin{aligned} \delta r &= 0 \text{ for } r = 0 \\ \delta p &= 0 \text{ for } r = R \end{aligned}\right\} \tag{20}$$

We write the differential equation (17) as

$$y\frac{d^2\xi}{dy^2}+\frac{d\xi}{dy}+\left[E-Fy-\frac{By^2}{2!}-\frac{By^3}{3!}-\ldots-\frac{By^n}{n!}-\ldots\right]\xi=0, \tag{21}$$

where

$$E=A-B \text{ and } F=1+B \tag{22}$$

Attempting a solution of the equation (21) in the form

$$\xi=\sum_{n=0}^{\infty} c_n y^{n+\alpha}$$
$$=c_0 y^{\alpha}+e_1 y^{\alpha+1}+\ldots+c_n y^{n+\alpha}+\ldots \tag{23}$$

at $y=0$, the indicial equation gives $\alpha^2=0$, *i.e.* the two roots are equal. with $\alpha=0$, (23) becomes

$$\xi=\sum_{0}^{\infty} c_n y^n \tag{24}$$

Substituting (24) in (21), and equating the coefficients of y^n.

We have the following recurrence formula :

$$(n+1)^2 c_{n+1}=-E c_n+F c_{n-1}+\frac{B}{2!}c_{n-2}$$
$$+\frac{B}{3!}c_{n-3}+\frac{B}{4!}c_{n-4}+\ldots+\frac{B}{n!}c_0$$

or

$$(n+1)^2 c_{n+1}+E c_n-F c_{n-1}.\ -B\sum_{\lambda=0}^{n-2}\frac{c_\lambda}{(n-\lambda)!}=0 \tag{25}$$

Dividing (25) by $(n+1)^2 c_n$, and proceeding to the limit as $n\to\infty$ we have

$$\lim\frac{c_{n+1}}{c_n}=0 \tag{26}$$

The equation (26) shows that the series (24) is convergent for $y=0$.

Thus for the given density relation radial pulsations are stable near the boundary of the cylinder.

The full solution of the equation (21) is[4]

$$\xi=c_0.(M+N\log y)\left[1+\frac{(-E)}{1^2}y+\left(\frac{(-E)^2}{1^2.2^2}+\frac{F}{2^2}\right)y^2\right.$$
$$\left.+\left(\frac{(-E)^3}{1^2.2^2.3^2}+(-E).F.\left(\frac{1}{1^2.3^2}+\frac{1}{2^2.3^2}\right)+\frac{B}{2!\,3^2}\right)y^3+\ldots\right]$$
$$-2c_0.N\left[\frac{(-E)}{1^3}y+\left(\frac{(-E)^2}{1^2.2^2}(1+\tfrac{1}{2})+\frac{F}{2^3}\right)y^2\right.$$
$$+\left(\frac{(-E)^3}{1^2.2^2.3^2}(1+\tfrac{1}{2}+\tfrac{1}{3})+\frac{(-E).F}{2^2.3^2}(\tfrac{1}{2}+\tfrac{1}{3})\right.$$
$$\left.\left.+\frac{(-E).F}{1^2.3^2}(1+\tfrac{1}{3})+\frac{B}{2!\,3^3}\right)y^3+\ldots\right] \tag{27}$$

M and N are constants.

REFERENCES

1. Chandrasekhar, S. and Fermi, F. *Ap. J.*, **118** : 116, (1953).
2. Eznar Lyttkens. *Ap. J.*, **119** : 413, (1954).
3. Bhatnagar, P. L. *Z. Astrophs*, 43 : 273, (1957).
4. (*i*) Bromwich, T. J. "An introduction to the theory of Infinite Series", 38, (1926).
   (*ii*) Whittaker and Watson : "A Course in Modern Analysis", 197–198, (1927).
   (*iii*) Levy, H. and Baggett. E. A. : "Numerical Solutions of Differential Equations", 84–87.

STABILITY OF COMPLEX DIFFERENCE EQUATIONS

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ABSTRACT

Lyapunov functions have been widely used in the "Stability Theory of Differential Equations." Making use of Massera's method of constructing a Lyapunov function, the author has studied the Stability properties of Complex Difference Equations.

INTRODUCTION

Lyapunov functions have been widely used in the "Stability Theory of Differential Equations" Massera's method of constructing a Lyapunov functions is considered to be the simplest one. Halanay has used Lyapunov functions to stability properties of real difference equations containing real functions. In this paper we extend this method to study the stability properties of complex difference equations. We consider the difference equations

$$x(z_{k+1}) = f(z_k, x(z_k))$$

$$y(z_{k+1}) = g(z_k, y(z_k)) \qquad \text{for} \qquad k \quad 0, 1, 2,$$

where x, y, f and g can be scalars or vectors, z_k's are complex variables, all distinct and $f(z_k, 0) \equiv 0$ $g(z_k, 0) \equiv 0$. f and g are regular analytic in x and y respectively.

Suppose $x(z_k, z_p, x_p)$ is the solution of (1) for $k \geqq p$ with $x(z_p, z_p, x_p) = x_p$ $x(z_p) = x_p$. Similarly $y(z_k, z_p, y_p)$ is the solution of (2) for $k \geqq p$ with $y(z_p, z_p, y_p) = y_p$. k and p are non-negative integers.

First we give the following definitions of uniform stability and uniform asymptotic stability.

(*i*) Equation (1) is said to be uniformly stable with respect to equation (2) if given any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for any $p \geqq 0$

$$| x_p - y_p | < \delta(\varepsilon) \text{ implies}$$

$$| x(z_k, z_p, x_p) - y(z_k, z_p, y_p) | < \varepsilon \text{ for all } k \geqq p.$$

(*ii*) Equation (1) is said to be uniformly asymptotically stable with respect to equation (2) if condition (*i*) holds and if for any $\varepsilon > 0$ there exists $N(\varepsilon)$ and δ_0 such that for any $p \geqq 0$

$$| x(z_k, z_p, x_p) - y(z_k, z_p, y_p) | < \varepsilon$$

whenever $| x_p - y_p | < \delta_0$ and $k \geqq p + N(\varepsilon)$.

We may also assume that $f(z_k, x)$ and $g(z_k, y)$ satisfies a Lipschitz condition that is

(*iii*) $| f(z_k, x) - f(z_k, \bar{x}) | \leqq L_r (x - \bar{x})$ for all $k \geqq 0$, $| x | \leqq r$, $| \bar{x} | \leqq r$

and

(iv) $| g(z_k, y) - g(z_k, \bar{y}) | \leq k_r | y - \bar{y} |$ for $| y | \leq r$, $| \bar{y} | \leq r$.

Theorem 1.—Suppose equation (1) is uniformly asymptotically stable with respect to equation (2). Let the functions $f(z_k, x)$ and $g(z_k, y)$ satisfy the Lipschitz condition. Then there exists a Lyapunov functions $V(t_k, x, y)$ for $k \geq 0$, $| z_k | = t_k$ and $| x-y | < \delta$ (δ_0) which satisfy the following conditions.

$$(a) \quad a(| x - y |) \leq V(t_k, x, y) \leq b(| x - y |)$$

$$(b) \quad V(t_{k+1}, x(z_{k+1}, z_k, x), y(z_{k+1}, z_k, y)) - V(t_k, x, y)$$

$$\leq - C (| x(z_{k+1} ; z_k ; x) - y (z_{k+1} ; z_k ; y |)$$

$$(c) \quad | V (t_k, x, y) - V (t_k, \bar{x}, \bar{y}) | \leq M [| x - \bar{x} | + | y - \bar{y} |]$$

where $a(r)$, $b(r)$ and $c(r)$ are continuous, strictly increasing functions for $a(o) = b(o) = 0$ and $c(o) = 0$ and M is a constant.

Here the Lyapunov function V is a function of the real variables.

Proof of theorem 1.—Suppose $G(r)$ is a function satisfying the conditions mentioned in[1], that is $G(r)$ is a definite function for $r > 0$

$$G(o) = 0, \ G'(o) = 0, \ G'(r) > 0, \ G''(r) > 0.$$

$$\text{For } \alpha > 1, \ G\left(\frac{r}{\alpha}\right) < \frac{1}{\alpha} \cdot G(r)$$

As in[2], suppose

$$V(t_k ; x ; y) = \sup_{l \geq 0} G(| x (z_{k+l}, z_k, x_k) - y(z_{k+l}, z_k, y_k) |) \frac{1+al}{1+l}$$

where k, l are positive integers. Following the same line of argument as in[1] and[2] we can prove that the Lyapunov function $V(t_k, x, y)$ satisfies the required conditions. The proof of theorem 1 is complete.

§2. We now consider the complex difference equations

$$(3) \qquad x(z_{k+1}) = f(z_k, x(z_k)) + F_1(z_k, x(z_k))$$

$$(4) \qquad y(z_{k+1}) = g(z_k, x(z_k) + F_2 (z_k, y(z_k))$$

which correspond to the perturbed equations in the case of real equations having real functions.

Let $\bar{x}(z_k, z_p, x_p)$ and $\bar{y}(z_k, z_p, y_p)$ be the solutions of (3) and (4) respectively.

(v) Equation (1) is said to be weakly stable with respect to equation (2) if for each $\varepsilon > 0$ there exists $\delta_1 = \delta_1(\varepsilon) > 0$ and $\delta_2 = \delta_2(\varepsilon) > 0$ such that the solutions $\bar{x}(z_k, z_p, x_p)$ and $\bar{y}(z_k, z_p, y_p)$ of (3) and (4) respectively satisfies the condition.

$$| \bar{x} (z_k ; z_p ; x_p) - y (z_k ; z_p ; y_p) | < \varepsilon \text{ for } k \geq p, \ p \geq 0 \text{ whenever}$$

$$| x_p - y_p | < \delta_1 \text{ and } | F_1 (z_k, x_k) | + | F_2 (z_k, y_k) | < \delta_2$$

Theorem 2.—If equation (1) is uniformly asymptotically stable with respect to equation (2), then equation (1) is weakly stable with respect to equation (2).

Proof of theorem 2.—The proof follows from the proof of a similar result proved in the case of real difference equation[1].

Remark.—I. G. Malkin has proved a theorem in the case of ordinary differential equations, that uniform asymptotic stability implies total stability. This result has been proved even in the case of delay differential equations[3], differential equations of finite time lags[4] and also functional differential equations in general. What happens if that uniformity is not there? is still an open question.

REFERENCES

1. Halanay, A. Quelques questions de la theorie de la stabilite pour les systemes aux differences finies. *Arch. Rat. Mech Analysis*. 12 : 150-154 (1963).
2. Driver, R. D Note on a paper of Halanay on stability for finite difference equations. *Arch Rat Mech. Anal*: 18: 241-243 (1965).
3. Krasovskii, N. N. Stability of Motion (English) *Stanford University Press, Stanford* (1964).
4. Halanay, A. Differential Equations (English) *Academic Press*. (1966).

MATHEMATICAL THEORY OF FREQUENCY MODULATION WITH DAMPED WAVES

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ABSTRACT

For the purpose of transmission of radio signals, an electromagnetic wave may be modulated by amplitude, frequency or phase modulating methods. The case of modulation with damped waves has been previously discussed by the authors and it has been shown that the number of sidebands in a frequency—modulated wave depends on the nature of the modulated wave. It has been further shown that in such analysis positive zeros of transcendental functions are involved.

The present communication deals with further work on the subject by bringing into consideration the positive zeros of the transcendental function involved in the expression for the frequency modulation with damped sinusoidal wave when we retain second order terms in '*t*'. The mathematical analysis indicates the various values of the modulation index for which the sidebands become negligible.

INTRODUCTION

In one of our previous communications[1] we have discussed the mathematical aspect of the theory of frequency modulation with damped waves. It was shown that for modulation with damped sinusoidal waves when the damping factor tends to unity, the side-bands almost vanish for a particular value of the modulation index. It was further shown that this type of modulation involves the positive zeros of a transcendental function and there might be several values of the modulation index for which the side-bands may approximately vanish.

It may be mentioned that in the above analysis the expressions for the modulation wave $M(t)$ were discussed only upto the first power of 't', and in the present communication it has been extended to the second power of 't' and the various values of the modulation index for which the side-bands become negligible have been considered.

1. *General considerations*

In the case of frequency modulation with damped waves or short duration, it has been shown[1] that the expression for the modulated wave can be written as

$$M(t) = A_c \sum_{n=-\infty}^{\infty} [J_n(\beta) - \alpha t\beta\, J_n'(\beta)] \cos \{ (w_c + nw_v)\, t + \tfrac{1}{2}n\pi \}, \qquad (1\cdot1)$$

where the symbols have their usual meanings.

There we have discussed the case when $n = \pm 1$ subject to the condition that αt approximates to unity and shown that for the modulation index $\beta = 8{\cdot}417$, the side-bands for $n = \pm 1$ almost vanish. We have further shown that there may be several values of β for which the side-bands become negligible. Mathematical analysis shows that numerous values[2] of β may be obtained, depending upon the value of n and αt, which satisfy the transcendental equation

$$J_n(\beta) - \alpha\beta t\, J_n'(\beta) = 0.$$

In the present case we have retained terms upto t^2 in the expression for the modulated wave and thereby we get

$$M(t) = A_c \sum_{n=-\infty}^{\infty} \left[J_n(\beta) - \alpha\beta t J_n'(\beta) + \frac{\alpha^2 t^2}{\underline{|2}} \{\beta^2 J_n''(\beta) + \beta J_n'(\beta)\} \right] \times \cos\{(w_c + nw_v)t + \tfrac{1}{2}n\pi\} \tag{1·2}$$

A method has been shown to determine various values of the modulation index β for which the side-bands may be negligible corresponding to the above expression for $M(t)$.

The expression (1·2) may be cast into the form

$$M(t) = -A_c\, \alpha\, t \sum_{n=-\infty}^{\infty} \left\{ \left[n + \frac{\alpha\, t}{2}(\beta^2 - n^2) - \frac{1}{\alpha t} \right] J_n(\beta) - \beta J_{n+1}(\beta) \right\} \times \cos\{(w_c + nw_v)t + \tfrac{1}{2}n\pi\} \tag{1·3}$$

by virtue of the results

$$\beta^2 J_n''(\beta) + \beta J_n'(\beta) + (\beta^2 - n^2) J_n(\beta) = 0$$

and
$$\beta J_n'(\beta) = n J_n(\beta) - \beta J_{n+1}(\beta)$$

If we examine for what values of β, the sidebands almost die out, it is necessary to determine the real zeros of the function

$$\{f(z) + n\} J_n(z) - z J_{n+1}(z),$$

under the special case when $f(z) \equiv \left\{ n + \frac{\alpha t}{2}(z^2 - n^2) - \frac{1}{\alpha t} \right\}$

The numerical values of β have been computed by solving the equation

$$G_n(\beta) \equiv \left\{ n + \frac{\alpha t}{2}(\beta^2 - n^2) - \frac{1}{\alpha t} \right\} J_n(\beta) - \beta J_{n+1}(\beta) = 0,$$

for $n = 1, 2, 3, 4$ and $\alpha t = 1, 2, 10, \frac{1}{2}, \frac{1}{8}$;

by the aid of the method required to determine the real zeros of the function $G_n(\beta)$ associated with the even function

$$\left\{ \frac{\alpha t}{2}(\beta^2 - n^2) - \frac{1}{\alpha t} \right\}^{(3)}$$

2. *Determination of smallest zeros of $G_n(\beta)$*

It can be shown that σ – numbers corresponding to the function

$$G_n(\beta) \equiv \left\{ n + \sum_{m=0}^{l} A_m \beta^{2m} \right\} J_n(\beta) - \beta J_{n+1}(\beta)$$

are given by the relations

$$\sigma^{(1)}_{n, A_{i,1}} = \frac{A_0 + n + 2 - 4A_1(n+1)}{4(A_0 + n)(n+1)} \tag{2·1}$$

$$\sigma^{(2)}_{n, A_{i,1}} = \frac{(A_0 + n + 2)^2 + 4(n+1)}{2^4 (A_0 + n)^2 (n+1)^2 (n+2)} + \frac{A_1\{A_1 - 1/(n+1)\}}{(A_0 + n)^2} - \frac{2A_2}{A_0 + n} \tag{2·2}$$

$$\underset{n,\,A_{i},1}{\sigma^{(3)}} = \underset{n,\,A_{i},1}{\sigma^{(1)}}\left[\underset{n,A_{i},1}{\sigma^{(2)}} - \left\{\frac{A_0+n+4-8A_1\,(n+2)+32\,(n+1)\,(n+2)}{2^5(A_0+n)\,(n+2)\,(n+1)}\right\}\right]$$

$$+\frac{A_0+n+6}{2^7(A_0+n)\,(n+3)\,(n+2)\,(n+1)} - \frac{3A_1}{2^5(A_0+n)\,(n+2)\,(n+1)}$$

$$+\frac{3A_2}{2^2(A_0+n)\,(n+1)} - \frac{3A_3}{(A_0+n)} \tag{2·3}$$

Replacing $\sum_{m=0}^{l} A_m z^{2m}$ by $(z^2-n^2)\ \frac{at}{2} - \frac{1}{at}$ we get $A_0 = -\frac{atn^2}{2} - \frac{1}{at}$, $A_1 = \frac{at}{2}$ and $A_i = 0$ for all integral values of $i > 1$, and the relations (2·1), (2·2) and (2·3) reduce to the forms

$$\underset{n,A_0,A_1,1}{\sigma^{(1)}} = \frac{\{at(n+2)-1\}^2+1}{4\{(atn-1)^2+1\}\,(n+1)} \tag{2·4}$$

$$\underset{n,A_0,A_1,1}{\sigma^{(2)}} = \frac{\{(atn-1)^2+1-4at\}^2+16a^2t^2\,(n+1)}{2^4\{(atn-1)^2+1\}^2\,(n+1)^2\,(n+2)} + \frac{a^3t^3(at-2/(n+1))}{\{(atn-1)^2+1\}^2} \tag{2·5}$$

$$\underset{n,A_0,A_1,1}{\sigma^{(3)}}$$

$$=\underset{n,A_0,A_1,1}{\sigma^{(1)}}\left[\underset{n,A_0,A_1,1}{\sigma^{(2)}} - \frac{1}{8(n+2)}\left\{\underset{n,A_0,A_1,1}{\sigma^{(1)}} + \frac{at[at(n+3)-1]}{\{(atn-1)^2+1\}\,(n+1)}\right\}\right]$$

$$+\frac{(atn-1)^2+1-12at}{2^7\{(atn-1)^2+1\}\,(n+3)\,(n+2)\,(n+1)} + \frac{3a^2t^2}{2^5\{(atn-1)^2+1\}\,(n+2)\,(n+1)} \tag{2·6}$$

We shall now furnish the numerical analysis required for determining smallest zeros of $G_n(\beta)$ associated with the function $n + \frac{at}{2}(\beta^2-n^2) - \frac{1}{at}$ in the form of Tables I—V by means of the results (2·4), (2·5) and (2·6).

TABLE I

Smallest zero $g_{1,1,1}$ of $G_1(\beta)$ corresponding to $at = 1$.

r	$\left[\underset{1,\,A_0,\,A_1,1}{\sigma^{(r)}}\right]^{-\frac{1}{2r}}$	$\left[\underset{1,\,A_0,\,A_1,1}{\sigma^{(r)}} \Big/ \underset{1,\,A_0,\,A_1,1}{\sigma^{(r+1)}}\right]^{\frac{1}{2}}$	Smallest zero of $G_1(\beta)$
1	1·265	1·711	
2	1·471	1·539	$g_{1,1,1} = 1·5$
3	1·493	..	

TABLE II

Smallest zero $g_{2,1,1}$ of $G_2(\beta)$ corresponding to $at = 1$.

r	$\left[\underset{2,\,A_0,\,A_1,1}{\sigma^{(r)}}\right]^{-\frac{1}{2r}}$	$\left[\underset{2,\,A_0,\,A_1,1}{\sigma^{(r)}} \Big/ \underset{2,\,A_0,\,A_1,1}{\sigma^{(r+1)}}\right]^{\frac{1}{2}}$	Smallest zero of $G_2(\beta)$
1	1·549	1·983	
2	1·753	1·784	$g_{2,1,1} = 1·77$
3	1·763	..	

TABLE III

Smallest zero $g_{3,1,1}$ of $G_3(\beta)$.

r	$\left[{}^{\sigma}3, \overset{(r)}{A_0}, A_{1,1}\right]^{-\frac{1}{2r}}$	$\left[{}^{\sigma}3, \overset{(r)}{A_0}, A_{1,1} \Big/ {}^{\sigma}3, \overset{(r+1)}{A_0}, A_{1,1}\right]^{\frac{1}{2}}$	Smallest zero of $G_3(\beta)$
1	2·17	3·106	
2	2·595	2·679	$g_{3,1,1} = 2{\cdot}64$
3	2·623	. .	

TABLE IV

Smallest zero $g_{4,\cdot,1}$ of $G_4(\beta)$.

r	$\left[{}^{\sigma}4, \overset{(r)}{A_0}, A_{1,1}\right]^{-\frac{1}{2r}}$	$\left[{}^{\sigma}4, \overset{(r)}{A_0}, A_{1,1} \Big/ {}^{\sigma}4, \overset{(r+1)}{A_0}, A_{1,1}\right]^{\frac{1}{2}}$	Smallest zero of $G_4(\beta)$
1	2·773	4·478	
2	3·524	3·71	$g_{4,1,1} = 3{\cdot}6$
3	3·585	. .	

Repeating the process as given by Tables I – IV, we can obtain the smallest zeros of $G_n(\beta)$ corresponding to $at = 2, 10, \frac{1}{2}, \frac{1}{3}$, as incorporated in Table V.

TABLE V

First zeros $g_{n,at,1}$ of $G_n(\beta) \equiv \left\{n + \frac{at}{2}(\beta^2 - n^2) - \frac{1}{at}\right\} J_n(\beta) - \beta J_{n+1}(\beta)$ for $n = 1,2,3,4$

n	$g_{n,1,1}$	$g_{n,2,1}$	$g_{n,10,1}$	$g_{n,\frac{1}{2},1}$	$g_{n,\frac{1}{3},1}$
1	1·5	0·82	0·93	. .	. .
2	1·77	1·75	1·938	. .	. .
3	2·64	2·75	2·944	3·6	. .
4	3·6	3·76	3·9475	3·86	. .

SUMMARY AND CONCLUSIONS

We have seen already[2] that the first zeros $j_{n,at,1}$ of

$F_n(\beta) \equiv \left(n - \frac{1}{at}\right) J_n(\beta) - \beta J_{n+1}(\beta)$ for $n = 1, 2, 3, 4$ are given by

TABLE VI

n	$j_{n,1,1}$	$j_{n,2,1}$	$j_{n,10,1}$	$j_{n,\frac{1}{2},1}$	$j_{n,\frac{1}{3},1}$
1	5·1356	1·3565	1·7352	. .	. .
2	2·2425	2·6544	2·8459	6·38	. .
3	3·6138	3·9421	4·156	2·6501	7·6
4	4·8248	5·1009	5·2934	4·12	3·003

It is remarkable to note in this connection that the first zeros corresponding to the cases $n = 1$; $\alpha t = \frac{1}{2}, \frac{1}{3}$ and $n = 2$; $\alpha t = \frac{1}{3}$ for the functions $G_n(\beta)$ in Table V and $F_n(\beta)$ in Table VI are not real. On the other hand we observe that the first zeros of $F_n(\beta)$ corresponding to the cases $n = 2, 3$; $\alpha t = \frac{1}{2}, \frac{1}{3}$ are real of magnitudes 6·38 and 7·6 respectively, whereas the first zeros of $G_n(\beta)$ corresponding to the same cases are not real.

It is interesting to note that the first zeros of $G_n(\beta)$ are less than the index number n in cases when $\alpha t = \frac{1}{2}, 1, 2, 10$ except when $n = 1$ and $\alpha t = 1$ and when $n = 3$ and $\alpha t = \frac{1}{2}$. It might be further observed that the values of β in the cases when $\alpha t = 10$ are very nearly equal to the index number n ; whereas the zeros of $F_n(\beta)$ behave in rather an erratic manner.

It may be concluded that there may be a large number of sidebands in frequency modulation with damped waves, but many of them may vanish for various values of modulation index and consequently, reduce the distributing effect of such modulation.

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REFERENCES

1. Mukherjee, S. R. Theory of modulation with damped waves. *Proc. Nat. Acad. Sci (India) Sec. A* 35 : (II), 113-120, (1965).
2. Mukherjee, S. R. and Bhowmick, K. N. Theory of Frequency modulation with damped waves. *Accepted for publication in India Jour. of Pure and Applied Physics*, (1966).
3. Mukherjee, S R. and Bhowmick, K. N. On zeros of a transcendental function associated with Bessel functions of the first kind of orders ν and $\nu + 1$; (Part II). *Accepted for publication in Proc. Nat. Acad. Sci, (India), Sec. A*, 36 : (1966).

STUDIES IN THE SYNTHESIS OF 3-(SUBSTITUTED PHENYL)-1-BIPHENYLYL-2-PROPEN-1-ONES OF BIO-CHEMICAL IMPORTANCE

By

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ABSTRACT

This paper deals with the synthesis of eleven new 3-(Substituted phenyl)-1-biphenylyl-2-propen-1-ones by Claisen-Schmidt condensation. The 2 : 4-dinitro phenyl-hydrazones of some of these compounds were also prepared.

INTRODUCTION

Jenney *et al*[1] have reported that the compounds containing the grouping -C=C - CO - and the phenyl-hydrazones of some carbonyl compounds exhibited strong toxic action towards rats, though the latter were not so effective as the former. The wide spread antibacterial properties encountered in the naturally occurring antibiotics which invariably contain the above grouping, led Marrian *et al*[2] to prepare some amino-analogues of the above type. The biological importance of such analogues has been supported by the researches of Minorn *et al*[3] and also some other workers[4-7]. With this in view, it was considered of interest to prepare some analogues of the above type and their 2 : 4-dinitro-phenyl hydrazones.

These compounds have been synthesised by condensing 4-aceto-biphenylyl with p-dimethyl-amino-benzaldehyde ; m-,p-nitro-benzaladehydes ; p-tolualdehyde ; p-methoxy-benzaldehydes ; β-resorcyldehyde ; o-vanillin, vanillin ; o-veratral-dehyde, veratral-dehyde and piperonal (Table 1).

TABLE 1

Analytical data of 3-(Substituted phenyl)-1-biphenylyl-2-propen-1-ones and their 2 : 4 dinitrophenyl hydrazones

Substituents	M. P. °C	Yield %	Formula	% Found C	% Found H	% Calculated C	% Calculated H	D.N.P. % N_2 Fcund	D.N.P. % N_2 Calc.	M P. °C
3 -Nitro-	164	60	$C_{21}H_{15}O_3N$	76·70	4·65	76·60	4·56	13·50	13·8	214
4-Nitro-	207	78	$C_{21}H_{15}O_3N$	76·50	4·73	76·60	4·56	13·6	13·8	251
4-Methyl-	187	73	$C_{22}H_{18}O$	88·60	6·20	88·59	6·04	11·5	11·7	21i
4-Methoxy-	138	72	$C_{22}H_{18}O_2$	83·90	5·95	84·08	5·73	11·1	11·3	209
4-Dimethyl-amino-	151	76	$C_{23}H_{21}ON$	84·36	6 38	84·40	6·42	13 7	13·8	206
2 : 4-Dihydroxy-	111	45	$C_{21}H_{16}O_3$	79·71	5 16	79·75	5·06	11·2	11 3	231
2-Hydroxy-3-methoxy-	151 (decomp)	44	$C_{22}H_{13}O_3$	79·81	5·69	80·00	5·46	-	-	-
3-Methoxy-4-hydroxy-	118	40	,,	80·18	5·23	80·00	5·46	-	-	-
2 : 3-Dimethoxy-	99	75	$C_{23}H_{20}O_3$	80·00	5·90	80·23	5·81	10·4	10·7	200
3 : 4-Dimethoxy	124	74	,,	80·40	5·61	80·23	5·81	10·5	10·7	178
3 : 4-Methylene-dioxy	195 196	78	$C_{22}H_{16}O_3$	80·61	4·63	80·49	4·87	10·9	11·0	201

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EXPERIMENTAL

General procedure : To a solution of 4-aceto-biphenyl (0·01 mole) and aryl-aldehyde (0·01 mole) in 25 ml aldehyde free ethanol, was added a strong solution of potassium hydroxide (5 g. in 5 cc of water) while shaking the ingredients. The mixture was kept at 17°C for half an hour and finally at room temperature for 36 hours. It was then neutralized with 5% hydrochloric acid, filtered and washed well with water. The acid of corresponding aldehyde formed during condensation was removed by washing its ethereal solution with 5% sodium bicarbonate. The ethereal solution was dried over anhydrous sodium sulphate, distilled and the product was crystallized.

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REFERENCES

1. Endre Jeney. *Tibor Zsolnai and Josef Lazar, Zentr. Bakteriol Parasitenk, Abt. I, Org.*, **163** : 291–301, (1955).
2. Marrian, *et al.*, *J. C. Soc.*, 1419, (1947).
3. Minorn, *et. al.*, *J. Agrl. Chem. Soc., Japan*, **28** : 791, (1954).
4. Eton, J. K. and Davies, R. G. *Ann. Applied Biol.*, **37** : 471–89, (1950).
5. Clark, S. F. *U. S*, 2805184, Sept 3, (1957).
6. Dinkar, *et al.* *Labdev. J. Sci and Tech.*, **2** : 66, (1964).
7. Dinkar, S. C. Kushwaha and Lal, J. B. *Proc. Nat. Acad. Sci., India*, **34** : 450, (1964).

EXISTENCE AND UNIQUENESS OF FLOWS BEHIND THREE-DIMENSIONAL STATIONARY CURVED MAGNETOGAS-DYNAMIC SHOCK WAVES

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ABSTRACT

The purpose of this paper is to show that the integration of the various conservation equations is equivalent to the solution of a Cauchy problem with the shock front as surface on which the initial data is given. Using Cauchy-Kowaleski theorem, the condition which must be satisfied in order that there may exist a flow behind in the neighbourhood of the shock, has been obtained in conducting fluids.

1. INTRODUCTION

Pant and Mishra[1] obtained the conditions for the existence and uniqueness of flows behind three-dimensional stationary and pseudo-stationary magnetogas-dynamic shock waves by using the Cauchy-Kowaleski theorem. They applied the existence theorem after reducing the basic equations to the 'normal' form by making use of a new coordinate system in which for stationary (pseudo-stationary) flows, a stream-line (streak-line) is taken as one of the coordinate curves.

In this paper we have taken a magnetic-line as one of the coordinate curves in the new coordinate system to find out the condition for existence and uniqueness of flows behind three-dimensional stationary curved shock waves. By making use of this coordinate system and following the method of present paper, the Cauchy-Kowaleski conditions can be easily obtained for flows behind pseudo-stationary and unsteady curved magnetogasdynamic shock waves. For unsteady flows we are required to express the equations of motion in a coordinate system moving with the shock wave. This has been done by the authors elsewhere[2].

2 BASIC EQUATIONS AND SHOCK CONDITIONS

If viscosity, thermal conduction and electrical resistance are absent, the equations governig the three-dimensional steady motion of a continuous conducting gas are[3]

$$-H_j \partial_j u_i + u_j \partial_j H_i + H_i \partial_k u_k = 0, \tag{2·1}$$

$$\rho u_j \partial_j u_i + \partial_i p - \frac{1}{4\pi} H_j \partial_j H_i + \frac{1}{4\pi} H_k \partial_i H_k = 0, \tag{2·2}$$

$$u_i \partial_i \rho + \rho \partial_i u_i = 0, \tag{2·3}$$

$$\partial_i H_i = 0, \tag{2·4}$$

$$u_i \partial_i \eta = 0, \tag{2·5}$$

where H_i stand for the components of the magnetic field, u_i for the fluid velocity components, p for the pressure, ρ for the density and η for the specific entropy of the fluid. The above equations are referred to a rectangular Cartesian coordinate system x_i $(i = 1, 2, 3,)$ and ∂_i denotes partial derivative with respect to x_i. Since there is no distinction between covarient and contravarient indices within a rectangular system, we may write an index as a superscript or subscript without modification of the value of the term in which the index occurs. Also it is to be understood in the above and in the following discusson, unless the contrary is stated, that an index which occurs twice in a term is to be summed over the admissable values of the index.

In addition, the equation of state for a perfect gas is

$$p = \exp(\eta / JC_v)\, \rho^{\gamma}, \tag{2·6}$$

where J is the mechanical equivalent of heat and γ is the ratio of two specific heats C_p and C_v assumed constant.

By differentiating (2·6) and using (2·5) we obtain

$$u_i\, \partial_i p = c^2\, u_i\, \partial_i \rho, \tag{2·7}$$

where $c^2 = \gamma p/\rho$, is the speed of sound.

A quantity f if evaluated infront of the shock surface will be denoted by f_1; if in the region behind the shock surface, then by f. The jump in f across the shock surface is expressed by

$$[f] \equiv f - f_1. \tag{2·8}$$

The expressions for flow and field quantities just behind the shock separately in terms of their values just infront of the shock are given by the relations

$$[\mathring{H}_i] = S_H(\mathring{H}_{1i} - \mathring{H}_{1n}\, \mathring{X}_i), \tag{2·9}$$

$$[\mathring{U}_i] = \frac{\mathring{U}_{1n}}{\mathring{H}_{1n}}\, \mathring{A}_1\, S_H\, \mathring{H}_{1i} - \mathring{U}_{1n}\, \frac{(1+\mathring{A}_1\, S_H)}{1+S_H}\, S_H\, \mathring{X}_i\,, \tag{2·10}$$

$$[\mathring{p}] = S_H \frac{(1-\mathring{A}_1)}{1+S_H}\, \mathring{\rho}_1\, \mathring{u}^2_{1n} - \frac{1}{8\pi}\, S_H\,(2+S_H)\, \mathring{H}_{1\alpha}\, \mathring{H}_1^{\alpha}, \tag{2·11}$$

$$[\mathring{\rho}] = \frac{\mathring{\rho}_1(1-\mathring{A}_1)\, S_H}{1+\mathring{A}_1 S_H}\,, \tag{2·12}$$

where S_H is the magnetic field strength of the shock defined as

$$S_H\, \mathring{H}_{1\alpha} = [\mathring{H}_\alpha], \tag{2·13}$$

and

$$\mathring{A}_1 = \frac{\mathring{H}^2_{1n}}{4\pi \mathring{\rho}_1\, \mathring{u}^2_{1n}} < 1, \tag{2·14a}$$

$$\mathring{H}_{1n} = \mathring{H}_{1i}\, \mathring{X}_i\,;\; \mathring{u}_{1n} = \mathring{u}_{1i}\, \mathring{X}_i\,, \tag{2·14b}$$

$$\mathring{H}_{1\alpha} = \mathring{H}_1{}^{\beta}\, \mathring{a}_{\alpha} = H_{1i}\, \mathring{\partial}_{\alpha}\, \mathring{x}_{i \cdot \beta} \tag{2·14c}$$

Here $\overset{\circ}{x}_i$ are the rectangular Cartesian coordinates of a point on the shock surface, $\overset{\circ}{X}_i$ denote the components of the unit vector normal to the shock surface and $\overset{\circ}{a}_{\alpha\beta} \equiv \partial_\alpha \overset{\circ}{x}_i \, \partial_\beta \overset{\circ}{x}_i$ are the components of the first fundamental form of the shock surface where $\partial_\alpha \overset{\circ}{x}_i \equiv \partial \overset{\circ}{x}_i / \partial y^\alpha$; y^α being the Gaussian cordinates on the shock surface. In the above and in what follows the kernel letter ° denotes the quantity at the shock surface.

For a perfect gas, S_H is given by the relation

$$\overset{\circ}{C}_1{}^2 \, (\overset{\circ}{A}_1 - 1) = \frac{(\overset{\circ}{A}_1 - 1) \, \overset{\circ}{u}_1{}^2{}_n}{2(1+S_H)} \, (2 + \overset{\circ}{A}_1 S_H + \gamma \overset{\circ}{A}_1 S_H - \gamma S_H + S_H) \tag{2·15}$$

$$+ \frac{\overset{\circ}{A}_1 \overset{\circ}{u}_1{}^2{}_n}{2 \overset{\circ}{H}_1{}^2{}_n} \overset{\circ}{H}_{1\alpha} \overset{\circ}{H}_1{}^\alpha \, (2 + \overset{\circ}{A}_1 S_H + 2 S_H + \overset{\circ}{A}_1 S_H{}^2 + \gamma \overset{\circ}{A}_1 S_H - \gamma S_H).$$

If we assume that the flow and field infront of the shock are uniform then the values of $\partial_\alpha \overset{\circ}{H}_i$, $\partial_\alpha \overset{\circ}{u}_i$, $\partial_\alpha \overset{\circ}{p}$ and $\partial_\alpha \overset{\circ}{\rho}$ are given by the following relations.

$$\partial_\alpha \overset{\circ}{H}_i = - S_H (\overset{\circ}{X}_i \, \partial_\alpha \overset{\circ}{H}_{1n} + \overset{\circ}{H}_{1n} \, \partial_\alpha \overset{\circ}{X}_i) + \overset{\circ}{H}_1{}^\beta \, \partial_\beta \overset{\circ}{x}_i \, \partial_\alpha S_H, \tag{2·16}$$

$$\partial_\alpha \overset{\circ}{u}_i = \left(\frac{\overset{\circ}{u}_{1n}}{\overset{\circ}{H}_{1n}} \overset{\circ}{A}_1 \overset{\circ}{H}_1{}^\beta \, \partial_\beta \overset{\circ}{x}_i - \frac{\overset{\circ}{u}_{1n} (1 - \overset{\circ}{A}_1)}{(1 + S_H)^2} \overset{\circ}{X}_i \right) \partial_\alpha S_H \tag{2·17}$$

$$+ \left(\frac{S_H}{4\pi \, \overset{\circ}{\rho}_1 \overset{\circ}{u}_{1n}} \overset{\circ}{H}_1{}^\beta \, \partial_\beta \overset{\circ}{x}_i + \frac{S_H \overset{\circ}{H}_{1n}}{4\pi \, \overset{\circ}{\rho}_1 \overset{\circ}{u}_{1n}} \frac{(1 - S_H)}{(1+S_H)} \overset{\circ}{X}_i \right) \partial_\alpha \overset{\circ}{H}_{1n},$$

$$+ \left(- \frac{\overset{\circ}{A}_1}{\overset{\circ}{H}_{1n}} S_H \overset{\circ}{H}_1{}^\beta \, \partial_\beta \overset{\circ}{x}_1 - \frac{S_H}{1+S_H} (1 + \overset{\circ}{A}_1) \overset{\circ}{X}_i \right) \partial_\alpha \overset{\circ}{u}_{1n} - S_H \overset{\circ}{u}_n \, \partial_\alpha \overset{\circ}{X}_i,$$

$$\partial_\alpha \overset{\circ}{p} = \left(\frac{(1 - \overset{\circ}{A}_1)}{(1+S_H)^2} \overset{\circ}{\rho}_1 \overset{\circ}{u}_{1n}{}^2 - \frac{1}{4\pi} (1+S_H) \overset{\circ}{H}_{1\delta} \overset{\circ}{H}_1{}^\delta \right) \partial_\alpha S_H \tag{2·18}$$

$$+ \frac{2 S_H}{1+S_H} \overset{\circ}{\rho}_1 \overset{\circ}{u}_{1n} \, \partial_\alpha \overset{\circ}{u}_{1n} + \frac{1}{4\pi} \overset{\circ}{H}_{1n} S_H{}^2 \frac{(3+S_H)}{(1+S_H)} \partial_\alpha \overset{\circ}{H}_{1n},$$

$$\partial_\alpha \overset{\circ}{\rho} = \frac{\overset{\circ}{\rho}_1 (1 - \overset{\circ}{A}_1)}{(1 + \overset{\circ}{A}_1 S_H)^2} \partial_\alpha S_H \tag{2·19}$$

$$- \frac{2 \overset{\circ}{A}_1 S_H (1+S_H) \overset{\circ}{\rho}_1}{(1+\overset{\circ}{A}_1 S_H)^2} \left(\frac{1}{\overset{\circ}{H}_{1n}} \partial_\alpha \overset{\circ}{H}_{1n} - \frac{1}{\overset{\circ}{u}_{1n}} \partial_\alpha \overset{\circ}{u}_{1n} \right),$$

and the value of $\partial_\alpha S_H$ is given by

$$\Big\{ 2 \, \overset{\circ}{C}_1{}^2 (\overset{\circ}{A}_1 - 1) - (\overset{\circ}{A}_1 - 1)(\overset{\circ}{A}_1 + \gamma \overset{\circ}{A}_1 - \gamma + 1) \, \overset{\circ}{u}_1{}^2{}_n - (3 \overset{\circ}{A}_1 S_H{}^2 \cdots \tag{2·20}$$

$$+ (4\overset{\circ}{A}_1+4+2\gamma\overset{\circ}{A}_1-2\gamma)\,S_H+\overset{\circ}{A}_1+\gamma\overset{\circ}{A}_1-\gamma+4)\,\frac{\overset{\circ}{H}_{1\delta}\,\overset{\circ}{H}_1{}^{\delta}}{4\pi\,\overset{\circ}{\rho}_1}\Big\}\,\partial_\alpha S_H$$

$$=\frac{\overset{\circ}{H}_{1n}}{2\pi\,\overset{\circ}{\rho}_1}\Big\{\frac{1}{4\pi\,\overset{\circ}{\rho}_1\,\overset{\circ}{u}_1{}^2{}_n}\,S_H(1+S_H)\,(\gamma+1+S_H)\,\overset{\circ}{H}_{1\delta}\,\overset{\circ}{H}_1{}^{\delta}-\frac{2\,\overset{\circ}{C}_1{}^2}{\overset{\circ}{u}_1{}^2{}_n}\,(1+S_H)$$

$$-\overset{\circ}{A}_1\,S_H{}^3+S_H{}^2(\gamma-2-2\overset{\circ}{A}_1-\gamma\,\overset{\circ}{A}_1)+S_H\,(\gamma\overset{\circ}{A}_1+\overset{\circ}{A}_1-4-\gamma)\Big\}\,\partial_\alpha\,\overset{\circ}{H}_{1n}$$

$$-2\,\overset{\circ}{u}_{1n}\Big\{\frac{\overset{\circ}{A}_1}{4\pi\,\overset{\circ}{\rho}_1\,\overset{\circ}{u}_1{}^2{}_n}\,S_H(1+S_H)\,(\gamma+1+S_H)\,\overset{\circ}{H}_{1\delta}\,\overset{\circ}{H}_1{}^{\delta}-\frac{2\overset{\circ}{C}_1{}^2\,\overset{\circ}{A}_1}{\overset{\circ}{u}_1{}^2{}_n}\,(1+S_H)$$

$$+S_H\,(\gamma\overset{\circ}{A}_1{}^2+\overset{\circ}{A}_1{}^2+1-\gamma)+2\Big\}\,\partial_\alpha\,\overset{\circ}{u}_{1n}\,.$$

where

$$\partial_\alpha\,\overset{\circ}{X}_i=-\overset{\circ}{a}{}^{\beta\gamma}\,\overset{\circ}{b}_{\beta\alpha}\,\partial_\gamma\,\overset{\circ}{x}_i, \tag{2·21}$$

$$\partial_\alpha\,\overset{\circ}{u}_{1n}=-\overset{\circ}{u}_{1\gamma}\,\overset{\circ}{a}{}^{\beta\gamma}\,\overset{\circ}{b}_{\beta\alpha}, \tag{2·22}$$

$$\partial_\alpha\,\overset{\circ}{H}_{1n}=-\overset{\circ}{H}_{1\gamma}\,\overset{\circ}{a}{}^{\beta\gamma}\,\overset{\circ}{b}_{\beta\alpha}. \tag{2·23}$$

Here $\overset{\circ}{b}_{\alpha\beta}$ are the components of the second fundamental form of the shock surface.

3 THE NEW COORDINATE SYSTEM

The shock configuration $\overset{\circ}{S}$ in a three-dimensional steady flow may be represented by the equations $\overset{\circ}{x}_i=\overset{\circ}{x}_i\,(y^1,y^2)$.

We assume that through each point of $\overset{\circ}{S}$, one and only one magnetic-line passes. At any point x_i on a magnetic-line behind the shock surface let ds be the elementary arc length along the magnetic-line ; there the components of the unit tangent vector to the magnetic-line are given by

$$\frac{\partial x_i}{\partial s}=\frac{H_i}{H}\ ;\ H^2=H_i\,H_i. \tag{3·1}$$

Further, let x_i be the cordinates of a point on the magnetic-line at a distance s from the shock surface in the region behind it. Through x_i, let us consider a surface S which is such that when $s\to 0$, S coincides with the shock surface $\overset{\circ}{S}$. Its equation is, then, given by Taub[4] as

$$x_i=x_i\,(y^1,y^2,s), \tag{3·2}$$

with the initial conditions.

$$x_i\,(y^1,y^2,0)=\overset{\circ}{x}_i\,(y^1,y^2),$$

and

$$X_i\,(y^1, y^2, 0) = \overset{\circ}{X}_i\,(y^1, y^2).$$

Since X_i and $\partial_\alpha x_i$ are three non-coplanar vectors, we can express u_i and H_i in terms of these by the relations

(3·3) $$u_i = u_n\, X_i + u^\alpha\, \partial_\alpha\, x_i,$$

(3·4) $$H_i = H_n\, X_i + H^\alpha\, \partial_\alpha\, x_i,$$

where

(3·5*a*) $$u_\alpha = u^\beta\, a_{\alpha\beta} = u_i\, \partial_\alpha\, x_i\,;\; H_\alpha = H_i\, \partial_\alpha\, x_i,$$

and

(3·5*b*) $$u_n = u_i\, X_i\,;\; H_n = H_i\, X_i.$$

At any point Q behind the shock surface, the flow and field variables are functions of y^1, y^2 and s, so in view of the above transformation we have

(3·6) $$\partial_j\, x_i = \delta_{ij} = \frac{H_i}{H}\, \partial_j s + \partial_\alpha\, x_i\, \partial_j\, y^\alpha,$$

and

(3·7) $$\partial_j f = \frac{\partial f}{\partial s}\, \partial_j\, s + \partial_\alpha f\, \partial_j\, y^\alpha.$$

Now multiplying (3·6) by X_i we obtain

(3·8) $$X_j = \frac{H_n}{H}\, \partial_j\, s.$$

Again multiplying (3·6) by $\varepsilon^{\gamma\beta}\, \varepsilon_{ilk}\, H_l\, \partial_\beta\, x_k$ and using the relations

(3·9) $$\varepsilon_{ilk}\, H_i\, H_l = 0,$$

and

(3·10) $$\varepsilon_{\alpha\beta}\, X_i = \varepsilon_{ijk}\, \partial_\alpha\, x_j\, \partial_\beta\, x_k,$$

we get

(3·11) $$\partial_j\, y^\alpha = \frac{1}{H_n}\, \varepsilon^{\alpha\beta}\, \varepsilon_{ijk}\, H_i\, \partial_\beta\, x_k.$$

Here $\varepsilon^{\alpha\beta}$ and ε_{ijk} are the components of the surface and space permutation tensors respectively.

Further, multiplying (3·11) by H_j and using (3·9) we get

(3·12) $$H_j\, \partial_j\, y^\alpha = 0.$$

Furthermore, multiplying (3·11) by X_j and using the relations

(3·13) $$\varepsilon_{ijk}\, X_i\, X_k = 0,$$

and

$$(3{\cdot}14) \qquad \varepsilon_{\alpha\beta} = \varepsilon_{ijk}\, X_i\, \partial_\alpha x_j\, \partial_\beta x^k,$$

we obtain

$$(3{\cdot}15) \qquad X_j\, \partial_j\, y^\alpha = -\frac{H^\alpha}{H_n}.$$

Also multiplying (3·11) by $\partial_\gamma x_j$ and using (3·10) we get

$$(3{\cdot}16) \qquad \partial_\gamma\, x_j\, \partial_j\, y^\alpha = \delta^\alpha{}_\gamma.$$

Multiplying (3·11) by u_j, using (3·3), (3·15) and (3·16) we obtain

$$(3{\cdot}17) \qquad u_j\, \partial_j\, y^\alpha = u^\alpha - \frac{u_n}{H_n} H^\alpha.$$

Now by virtue of the relations (3·7), (3·8), (3·12) the equations (2·1), (2·2), (2·3) and (2·7) assume respectively the forms

$$(3{\cdot}18) \qquad -H\frac{\partial u_i}{\partial s} + \frac{H}{H_n} H_i \frac{\partial u_j}{\partial s} X_j + \frac{H}{H_n} u_n \frac{\partial H_i}{\partial s} + u_j\, \partial_j\, y^\alpha\, \partial_\alpha\, H_i + H_i\, \partial_\alpha\, u_j\, \partial_j\, y^\alpha = 0,$$

$$(3{\cdot}19) \qquad \frac{\rho H}{H_n} u_n \frac{\partial u_i}{\partial s} + \frac{H}{4\pi H_n} H_k \frac{\partial H_k}{\partial s} X_i + \frac{H}{H_n} X_i \frac{\partial p}{\partial s} - \frac{H}{4\pi}\frac{\partial H_i}{\partial s}$$
$$+ \left(\partial_\alpha\, p + \frac{1}{4\pi} H_k\, \partial_\alpha\, H_k\right)\partial_i\, y^\alpha + \rho u_j\, \partial_j\, y^\alpha\, \partial_\alpha\, u_i = 0$$

$$(3{\cdot}20a) \qquad \frac{H}{H_n} u_n \frac{\partial \rho}{\partial s} + \frac{\rho H}{H_n} X_i \frac{\partial u_i}{\partial s} + \rho\, \partial_\alpha\, u_i\, \partial_i\, y^\alpha + u_i\, \partial_i\, y^\alpha\, \partial_\alpha\, \rho = 0.$$

$$(3{\cdot}21) \qquad \frac{\partial \rho}{\partial s} - \frac{1}{c^2}\frac{\partial p}{\partial s} + \frac{H_n}{H u_n} u_i\, \partial_i\, y^\alpha\, \partial_\alpha\, \rho - \frac{1}{c^2}\frac{H_n}{H u_n} u_i\, \partial_i\, y^\alpha\, \partial_\alpha\, p = 0.$$

Eliminating $\frac{\partial \rho}{\partial s}$ from the equation (3·20a), with the help of the equation (3·21) we obtain

$$(3{\cdot}20b) \qquad \frac{H}{H_n}\frac{u_n}{c^2}\frac{\partial p}{\partial s} + \frac{\rho H}{H_n} X_i \frac{\partial u_i}{\partial s} + \rho\, \partial_\alpha\, u_i\, \partial_i\, y^\alpha + \frac{1}{c^2} u_i\, \partial_i\, y^\alpha\, \partial_\alpha\, p = 0.$$

Multiplying the equation (3·19) by H_i we get

$$(3{\cdot}22) \qquad \frac{\rho H}{H_n} u_n\, H_i \frac{\partial u_i}{\partial s} = -\frac{H}{4\pi} H_k \frac{\partial H_k}{\partial s} - H\frac{\partial p}{\partial s} + \frac{H}{4\pi} H_i \frac{\partial H_i}{\partial s} - \rho u_j\, \partial_j\, y^\alpha\, H_i\, \partial_\alpha\, u_i.$$

Eliminating $X_j \frac{\partial u_j}{\partial s}$ from the equation (3·18) with the help of the equation (3·20b) we obtain

$$(3{\cdot}23)\quad \frac{H}{H_n} u_n \frac{\partial H_i}{\partial s} = H \frac{\partial u_i}{\partial s} + \frac{1}{\rho} \frac{H}{H_n} \frac{u_n}{c^2} \frac{\partial p}{\partial s} H_i + \frac{H_i}{\rho c^2} u_j \partial_j y^\alpha \partial_\alpha p - u_j \partial_j y^\alpha \partial_\alpha H_i.$$

Multiplying the equation (3·23) by H_i and eliminating $H_i \frac{\partial u_i}{\partial s}$ with the help of (3·22) we get

$$(3{\cdot}24)\quad \frac{H}{H_n} u_n H_i \frac{\partial H_i}{\partial s} = \left(-\frac{HH_n}{\rho u_n} + \frac{H^3 u_n}{\rho c^2 H_n} \right) \frac{\partial p}{\partial s} - \frac{H_n}{u_n} u_j \partial_j y^\alpha H_i \partial_\alpha u_i$$
$$+ \frac{H^2}{\rho c^2} u_j \partial_j y^\alpha \partial_\alpha p - u_j \partial_j y^\alpha H_i \partial_\alpha H_i.$$

Now, substituting in the equation (3·19) the values of $\frac{\partial H_i}{\partial s}$ and $H_i \frac{\partial H_i}{\partial s}$ from equations (3·23) and (3·24) we obtain

$$(3{\cdot}25)\quad \frac{\rho H u_n}{H_n}\left(1 - \frac{H^2_n}{4\pi \rho u^2_n}\right)\frac{\partial u_i}{\partial s} + \frac{H u_n}{H_n}\left\{ X_i \left(\frac{1}{u_n} + \frac{H^2}{4\pi \rho u_n c^2} - \frac{H^2_n}{4\pi \rho u^3_n} \right) \right.$$
$$\left. - \frac{H_n H_i}{4\pi \rho u_n c^2} \right\} \frac{\partial p}{\partial s} = \frac{1}{4\pi u_n} X_i u_j \partial_j y^\alpha \left(\frac{H_n}{u_n} H_k \partial_\alpha u_k - \frac{H^2}{\rho c^2} \partial_\alpha p + H_k \partial_\alpha H_k \right)$$
$$+ \frac{H_n}{4\pi u_n} u_j \partial_j y^\alpha \left(\frac{H_i}{\rho c^2} \partial_\alpha p - \partial_\alpha H_i \right) - \left(\partial_\alpha p + \frac{1}{4\pi} H_k \partial_\alpha H_k \right) \partial_i y^\alpha$$
$$- \rho u_j \partial_j y^\alpha \partial_\alpha u_i$$

Multiplying the equation (3·25) by X_i and substituting the value of $X_i \frac{\partial u_i}{\partial s}$ from the equation (3·20*b*), we obtain for $\frac{\partial p}{\partial s}$, the equation

$$(3{\cdot}26)\quad \frac{H u^2_n}{H_n} \left(\frac{1}{c^2} - \frac{1}{u^2_n} - \frac{H^2}{4\pi \rho c^2 u^2_n} + \frac{H^2_n}{4\pi \rho u^4_n} \right) \frac{\partial p}{\partial s} = \left(\frac{H^2_n}{4\pi \rho u^2_n} - 1 \right) \rho u_n \partial_\alpha u_i \partial_i y^\alpha$$
$$+ \frac{u_n}{c^2} \left(\frac{H^2}{4\pi \rho u^2_n} - 1 \right) u_i \partial_i y^\alpha \partial_\alpha p - \frac{1}{4\pi u_n} u_i \partial_i y^\alpha \left(\frac{H_n}{u_n} H_j \partial_\alpha u_j + H^\beta \partial_\beta x_j \partial_\alpha H_j \right)$$
$$+ \left(\partial_\alpha p + \frac{1}{4\pi} H_j \partial_\alpha H_j \right) \partial_i y^\alpha X_i + \rho u_i \partial_i y^\alpha \partial_\alpha u_j X_j.$$

Now if an analytic shock surface is given and if flow incident upon it is also given, then for the existence of the flow in the region behind the shock, we see, by the application of Cauchy-Kowaleski existence theorem, that the coefficient of $\partial p/\partial s$ in the equation (3·26) must now be zero. This shows that the coefficient of $\partial u_i/\partial s$ in the equation (3·25) should also not be zero*. From above considerations we obtain the following conditions :

*The equation (3·26) can be obtained only when the coefficient of $\partial u_i/\partial s$ in the equation (3·25) is not zero, otherwise we shall not be able to eliminate $X_i \partial u_i/\partial s$ between the equations (3·25) and (3·20*b*).

(3·27a) $$\left(1-\frac{c^2}{u^2_n}\right)\left(1-\frac{H^2_n}{4\pi\rho u^2_n}\right)\neq\frac{\dot{H}_\alpha\,\dot{H}^\alpha}{4\pi\,\rho\,u^2_n},$$

(3·27b) $$\left(1-\frac{H^2_n}{4\pi\,\rho\,u^2_n}\right)\neq 0,$$

and

(3·27c) $$u_n\neq 0,$$

which agree with the results obtained in reference[1].

REFERENCES

1. Pant, J. C. and Mishra. R. S. Exis ence and uniqueness of flows behind three-dimensional stationary and pseudo-stationary shock waves in couducting fluids *Proc. Nat Inst. Sci. India (in press)*
2. Sachdeva, S. K and Mishra, R. S. Determination of gradients of flow variables, vorticity and current density behind a two dimensional unsteady curved magnetogasdynamic shock, *Tensor, N S* 17 : 238–248, (1966).
3. Cowling. T. G. Magnetohydrodynamics, *Inter Science Pub.*, (1957).
4. Taub A. H. Determination of flows behind stationary and pseudo-stationary shocks, *Annals of Math.*, 62 : 300-325, (1955).

FLOW OF VISCO-ELASTIC MAXWELL FLUID THROUGH ELLIPTICAL TUBE

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§1. INTRODUCTION

The flow of steady viscous incompressible fluid through circular and elliptical tubes under the influence of a periodic pressure gradient has been investigated by Sexl[4] and Khamrui[2] respectively. Later, Drake[1] has discussed the flow of ordinary viscous liquid through rectangular channel. In the present paper the laminar flow of visco-elastic Maxwell fluid through uniform elliptical tube due to a periodic pressure gradient is studied. The solution is expressed in terms of Mathieu functions and results of two cases of very small and very large frequencies are obtained. The solution of the flow of visco-elastic liquid through circular cylinder, and the results of Sexl[4] and Khamrui[2] are deduced as particular cases of this investigation.

§2. BASIC EQUATIONS

The flow of a visco-elastic fluid of Maxwell type (*i.e.* a spring and a dash-pot arranged in series) is governed by the equations

$$\rho\left(\frac{\partial vi}{\partial t}+v^{i}{}_{,j}\ vi\right)=\tau^{ij}{}_{,j} \tag{1}$$

$$v^{i}{}_{,i}=0 \tag{2}$$

The stress-strain relations are

$$\left.\begin{aligned}\tau^{ij}&=-p\,g^{ij}+\tau'^{ij}\\ \left(1+x\frac{\partial}{\partial t}\right)\tau'^{ij}&=2\mu e^{ij}\\ e_{ij}&=\tfrac{1}{2}\left(v_{i,j}+v_{j,i}\right)\end{aligned}\right\} \tag{3}$$

Opertating $\left(1+x\dfrac{\partial}{\partial t}\right)$ and using (3), equation of motion (1) becomes

$$\left(1+x\frac{\partial}{\partial t}\right)\left(\frac{\partial v^{i}}{\partial t}+v^{i}{}_{,j}\,v^{j}\right)=-\frac{1}{\rho}\left(1+x\frac{\partial}{\partial t}\right)p_{,j}\,g^{ij}+2\nu\,e^{j}{}_{,j} \tag{4}$$

where τ^{ij} denotes the stress tensor, τ'^{ij} the deviatoric stress tensor, e_{ij} the strain rate of deformation tensor, g^{ij} the contravariant components of metric tensor, p the pressure, ρ the fluid density, x the relaxation time constant, μ the coefficient

of viscosity, ν the kinematic coefficient of viscosity, t the time v^i $(i=1, 2, 3)$ the component of velocity.

§3. FORMULATION AND SOLUTION OF THE PROBLEM

We shall consider the laminar flow of visco-elastic Maxwell fluid through uniform tube whose cross section is the ellipse, $x^2/a^2+y^2/b^2=1$, under the influence of periodic pressure gradient. Taking Z-axis along the axis of the tube, the velocity components are

$$v_1 = 0 \qquad v_2 = 0 \qquad v_3 = W(x, y, t) \tag{5}$$

Using the equations (1) – (3) and (5), equation (4) simplifies to

$$\left(1 + x\frac{\partial}{\partial t}\right)\frac{\partial W}{\partial t} = -\frac{1}{\rho}\left(1 + x\frac{\partial}{\partial t}\right)\frac{\partial p}{\partial^2 z} + \nu\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right) \tag{6}$$

Since the pressure gradient is periodic with period $2\pi/\omega$ we can write

$$-\frac{1}{\rho}\frac{\partial p}{\partial^2 z} = k\ Rl\ e^{i\omega t} \tag{7}$$

$$W = Rl\, f(x, y)e^{i\omega t} \tag{8}$$

where k is a real constant, ω is the frequency and Rl denotes the real part.

Using (7) and (8) in (6) we have

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + w\left(\frac{xw - i}{\nu}\right)(f + i\,k/w) = 0 \tag{9}$$

Now we shall obtain the solution of the problem with the boundary condition $W = 0$ at the cylinder

Changing (9) to elliptical coordinates ξ,η by the substitution

$$x+iy = c\cosh(\xi + i\eta) \qquad c = \sqrt{a^2 - b^2}$$

we obtain

$$\frac{\partial^2 F}{\partial \xi^2} + \frac{\partial^2 F}{\partial \eta^2} + \omega c^2\,\frac{(x\,\omega - i)}{\nu}\,(\cosh 2\xi - \cos 2\eta) = 0 \tag{10}$$

where we have taken

$$F = f + i\,k/\omega$$

On separating the variables of (10), we have

$$\frac{d^2\theta}{d\xi^2} - [\lambda - 2q\cosh 2\xi]\,\theta = 0 \tag{11}$$

$$\frac{d^2\psi}{d\eta^2} + [\lambda - 2q\cos 2\eta]\,\psi = 0 \tag{12}$$

where $\qquad F = \theta(\xi)\,\psi(\eta)$

and $$2q = \omega c^2 \frac{(x w - i)}{2\nu}$$

The fluid velocity is symmetrical about the axis of the ellipse and is single valued and periodic in η with π, so ψ is a multiple of the Mathieu functions[3] $c\,e_{2n}(\eta, q)$ of order $2n$ θ is then modified Mathieu function $c\,e_{2n}(\xi, q)$ we have

$$\left.\begin{aligned} F &= \sum_{0}^{\infty} c_{2n}\, c\,e_{2n}(\xi, q)\, c\,e_{2n}(\eta, q) \\ c\,e_{2n}(\xi, q) &= \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cosh 2r\,\xi \\ c\,e_{2n}(\eta, q) &= \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cos 2r\,\eta \end{aligned}\right\} \tag{13}$$

where the coefficients $A_{2r}^{(2n)}$ are functions of q.

Using the boundary condition

$$F = ik/\omega \qquad \text{when} \quad \xi = \xi_0$$

and employing the usual normalisation relations, we have

$$c_{2n} = \frac{ik}{\omega} A_{0}^{(2n)} \Big/ c\,e_{2n}(\xi_0, q) \tag{14}$$

We shall consider two cases of very small and large frequencies.

Case : (*a*) For very small frequencies $|q|$ is small. In that case, we can expand $c\,e_{2n}(\xi, q)$ and $c\,e_{2n}(\eta, q)$ in powers of q and retaining only first powers we get

$$A_{0}^{(0)} = 1\,, \quad A_{0}^{(2)} = \tfrac{1}{4} q + 0\,(q^3), \quad A_{0}^{(2n)} = 0\,(q^n) \tag{15}$$

Therefore

$$F = \frac{ik}{\omega}\left[1 + \tfrac{1}{2} q \,(\cosh 2\xi_0 - \cos 2\eta - \cosh 2\xi + \frac{\cosh 2\xi \cos 2\eta)}{\cosh 2\xi_0} \right] \tag{16}$$

or

$$f = \frac{ik}{2\omega} q \left(\cosh 2\xi_0 - \cos 2\eta - \cosh 2\xi + \frac{\cosh 2\xi \cos 2\eta}{\cosh 2\xi_0} \right)$$

which on using (8) gives the velocity of the fluid in this case as

$$W = Rl\left[\frac{ik}{8\nu} (x\omega - i)\,(\cosh 2\xi_0 - \cos 2\eta - \cosh 2\xi + \frac{\cosh 2\xi \cos 2\eta)}{\cosh 2\xi_0}\, e^{i\omega t} \right] \tag{17}$$

or

$$W = \frac{kc^2}{8\nu} (\cos \omega t - x\omega \sin \omega t)\,(\cosh 2\xi_0 - \cosh 2\xi - \cos 2\eta + \frac{\cosh 2\xi \cos 2\eta}{\cosh 2\xi_0})$$

The presence of x damps the motion of the fluid. Transforming into cartesian coordinates by

$$a = c \cosh \xi_0$$
$$b = c \sinh \xi_0$$

(17) can be expressed as

$$W = (k/2\nu) \quad \frac{a^2 b^2}{a^2 + b^2}\left(1 - x^2/a^2 - y^2/b^2\right) (\cos \omega t - x\omega \sin \omega t) \tag{18}$$

For $a = b$, we obtain from (18)

$$W = \frac{k}{4\nu} (a^2 - r^2) (\cos \omega t - x\omega \sin \omega t) \tag{19}$$

This is the corresponding solution for a circular cylinder.

When $x = 0$, from (18), we have

$$W = \frac{k}{4\nu} (a^2 - r^2) \cos \omega t \tag{20}$$

which is Sexl's[4] result.

From (19) and (20), it follows that the result of the visco-elastic fluid differs from that of an ordinary viscous fluid in the factor $(k/4\nu)$ $(\cos \omega t - x\omega \sin \omega t)$ in place of $(k/4\nu) \cos \omega t$, showing that the presence of relaxation time flattens the velocity profile.

We can also deduce the result of Khamrui[2] from (18) for an ordinary viscous liquid.

Case (*b*) For very large frequencies of pressure gradient, $|q|$ is very large. Then replacing $2q$ by $-2q'$, $q'>0$ and proceeding as before we have

$$\left.\begin{aligned} F &= \sum_{0}^{\infty} c_{2n}\, ce_{2n}(\xi, -q')\, ce_{2n}(\eta, -q') \\ ce_{2n}(\xi, -q') &= (-1)^n \sum_{r=0}^{\infty} (-1)^r \quad A_{2r}^{(2n)} \cosh 2r\xi \\ ce_{2n}(\eta, \quad q') &= (-1)^n \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} \cos 2r\eta \,. \end{aligned}\right\} \tag{21}$$

From the boundary condition and the normalisation relations and remembering that $A_0^{(0)} = 1$, $A_0^{(2n)}$ is small, when $n \geqslant 1$ (one). Thus we have the following asymptotic formula[3]

$$ce_0 \; (\xi, -q') \sim \left(\frac{2}{i \sinh\xi}\right)^{\frac{1}{2}} k_0 \cosh\left[\, 2\sqrt{q'} \cosh\xi - \tan^{-1}\left\{\tan\,(\pi' u - i\xi\tfrac{1}{2})\right\}\right] \tag{22}$$

where $$k_0 = \frac{ce_0(0)\, ce_0(\frac{1}{2}\pi)}{A_0^{(0)}\,(2\pi\sqrt{q'})^{\frac{1}{2}}}$$

when $q' >> 1$.

Therefore we have

$$F = \frac{ik}{\omega}\left(\frac{\cosh\frac{1}{2}\xi_0}{\cosh\frac{1}{2}\xi}\right)^{\frac{1}{2}} \exp\left[-2\sqrt{q'}\,(\cosh\xi_0 - \cosh\xi)\right]\, ce_0\,(\eta,-q') \qquad (23)$$

giving

$$f = \frac{ik}{\omega}\left[\,(\cosh\tfrac{1}{2}\xi_0/\cosh\tfrac{1}{2}\xi)^{\frac{1}{2}} \exp\left\{-2\sqrt{q'}\,(\cosh\xi_0 - \cosh\xi)\right\} ce_0\,(\eta,-q') - 1\right] \qquad (24)$$

The velocity of the fluid in this case is given by

$$W = Rl\frac{ik}{\omega}\left[\,(\cosh\tfrac{1}{2}\xi_0/\cosh\tfrac{1}{2}\xi)^{\frac{1}{2}} \exp\left\{-2\sqrt{q'}\,(\cosh\xi_0 - \cosh\xi)\right\}\right.$$

$$\left. ce_0\,(\eta,-q') - 1\right] e^{i\omega t} \qquad (25)$$

or,

$$W = \frac{k}{\omega}\sin\omega t + Rl\,\frac{ik}{\omega}\left[\,(\cosh\tfrac{1}{2}\xi_0/\cosh\tfrac{1}{2}\xi)^{\frac{1}{2}} \exp\left\{-2\sqrt{q'}\,(\cosh\xi_0 - \cosh\xi)\right\}\right.$$

$$\left. ce_0\,(\eta,-q')\right] e^{i\omega t}$$

In this case we see that the fluid motion has boundary layer character.

REFERENCES

1. Drake, D. G. *Qly. Journ. Mech. Appl. Maths.* **18**(1) : 1, (1965).
2. Khamrui. S. R. *Bull. Cal. Math. Soc* **49** : 57, (1957).
3. Maclachlan. N. W. *Theory and Applications of Mathieu Functions, Oxford.* (1947).
4. Sexl, Th. *Z. Phys.* **61** : 349, (1930).

INTEGRATION OF CERTAIN PRODUCTS INVOLVING A GENERALIZED MEIJER FUNCTION

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ABSTRACT

Recently, R. P Agarwal [cf. *Proc. Nat. Inst. Sci. India, Sec. A*, 31 (1965), pp. 536-546] gave a generalization of Meijer's G-function to two variables by means of a double Mellin-Barnes contour integral in the form*

$$G^{p, q, s, r, t}_{A, [C, E], B, [D, F]}\left[\begin{array}{c|c} x & (a) \\ & (c);(e) \\ & (b) \\ y & (d);(f) \end{array}\right]$$

$$= \frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} \Phi(\xi+\eta)\,\psi(\xi,\eta)\,x^{\xi}y^{\eta}\,d\xi\,d\eta,$$

where (a) denotes the sequence of A parameters

$$a_1, a_2, \ldots, a_A,$$

i.e. there are A of the a parameters, B of the b parameters and so on,

$$\Phi(\xi+\eta) = \frac{\prod_{j=1}^{p}\Gamma[1-a_j+\xi+\eta]}{\prod_{j=p+1}^{A}\Gamma[a_j-\xi-\eta]\prod_{j=1}^{B}\Gamma[b_j+\xi+\eta]}$$

$$\psi(\xi,\eta) = \frac{\prod_{j=1}^{q}\Gamma[c_j+\xi]\prod_{j=1}^{r}\Gamma[d_j-\xi]\prod_{j=1}^{s}\Gamma[e_j+\eta]\prod_{j=1}^{t}\Gamma[f_j-\eta]}{\prod_{j=q+1}^{C}\Gamma[1-c_j-\xi]\prod_{j=r+1}^{D}\Gamma[1-d_j+\xi]\prod_{j=s+1}^{E}\Gamma[1-e_j-\eta]\prod_{j=t+1}^{F}\Gamma[1-f_j+\eta]}$$

$$0\leqslant p\leqslant A,\ 0\leqslant q\leqslant C,\ 0\leqslant r\leqslant D,\ 0\leqslant s\leqslant E,\ 0\leqslant t\leqslant F,$$

the sequences of parameters

$a_1, a_2, \ldots, a_p$; $c_1, c_2, \ldots, c_q$; $d_1, d_2, \ldots, d_r$; $e_1, e_2, \ldots, e_s$ and $f_1, f_2, \ldots, f_t$ are such that none of the poles of the integrand coincide, and the paths of the integration are indented, if necessary, in such a manner that all the poles of

$\Gamma[d_j-\xi], j=1, 2, \ldots, r$ and $\Gamma[f_k-\eta], k=1, 2, \ldots, t$ lie to the right, and those of $\Gamma[c_j+\xi], j=1, 2, \ldots, q$, $\Gamma[e_k+\eta], k=1, 2, \ldots, s$ and $\Gamma[1-a_j+\xi+\eta], j=1, 2, \ldots, p$ lie to the left of the imaginary axis.

In the present paper we evaluate numerous integrals (both finite and infinite) containing certain products of the generalized Meijer function.

A number of known results are exhibited as special cases of our integrals.

*The notation for the G-function in two arguments that we discuss here differs markedly from the one given earlier by Professor Agarwal, though in essence the function remains the same.

1. INTRODUCTION

Recently, Sharma [10, pp. 26–40] defined the generalized Meijer function of two variables by means of a double contour integral in the form

$$(1\cdot1)\qquad S\left[\begin{array}{c|c|c} \begin{bmatrix} p & , & o \\ A & p & B \end{bmatrix} & (a)\,;\,(b) & \\ \begin{pmatrix} q & , & r \\ c-q & , & D-r \end{pmatrix} & (c)\,;\,(d) & x, y \\ \begin{pmatrix} k & , & l \\ E-k, & F-l \end{pmatrix} & (e)\,;\,(f) & \end{array}\right]$$

$$= \frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2} \frac{\prod\limits_{j=1}^{p}\Gamma(a_j+s+t)\prod\limits_{j=1}^{q}\Gamma(1-c_j+s)\prod\limits_{j=1}^{r}\Gamma(d_j-s)\prod\limits_{j=1}^{k}\Gamma(1-e_j+t)}{\prod\limits_{j=p+1}^{A}\Gamma(1-a_j-s-t)\prod\limits_{j=1}^{B}\Gamma(b_j+s+t)\prod\limits_{j=q+1}^{C}\Gamma(c_j-s)\prod\limits_{j=r+1}^{D}\Gamma(1-d_j+s)}$$

$$\times \frac{\prod\limits_{j=1}^{l}\Gamma(f_j-t)\,x^s y^t}{\prod\limits_{j=k+1}^{E}\Gamma(e_j-t)\prod\limits_{j=l+1}^{F}\Gamma(1-f_j+t)}\,ds\,dt,$$

where the contour L_1 in the s-plane runs from $-i\infty$ to $+i\infty$, curving if necessary so as to ensure that the poles of $\Gamma(d_j-s)$ $(j=1, 2, \ldots, r)$ lie to the right and the pole of $\Gamma(1-c_j+s)$ $(j=1, 2, \ldots, q)$ and $\Gamma(a_j+s+t)$ $(j=1, 2, \ldots, p)$ to the left of the contour. Similarly the contour L_2 in the t-plane consists of the portion of the imaginary axis from $-i\infty$ to $+i\infty$ along with the necessary loops so as to ensure that the poles of $\Gamma(f_j-t)$ $(j=1, 2, \ldots, l)$ lie to the right and the poles of $\Gamma(1-e_j+t)$ $(j=1, 2, \ldots, k)$ and $\Gamma(a_j+s+t)$ $(j=1, 2, \ldots, p)$ to the left of the contour.

The positive integers A, B, C etc. satisfy the following inequalities :—

$$D \geqslant 1, F \geqslant 1, A \geqslant 1, B \geqslant 1,$$

$$o \leqslant p \leqslant A,\ o \leqslant q \leqslant c,\ o \leqslant k \leqslant E,\ o \leqslant r \leqslant D,\ o \leqslant l \leqslant F,$$

$$A + C \leqslant B + D \text{ and } A + E \leqslant B + F.$$

The above integral converges under the following conditions

$$2(p+q+r) > A+B+C+D,\ |\arg.(x)| < (p+q+r-\tfrac{1}{2}A-\tfrac{1}{2}B-\tfrac{1}{2}C-\tfrac{1}{2}D)\pi,$$

$$2(p+k+l) > A+B+E+F,\ |\arg.(y)| < (p+k+l-\tfrac{1}{2}A-\tfrac{1}{2}B-\tfrac{1}{2}E-\tfrac{1}{2}F)\pi\,;$$

If $A+C=B+D$, $A+E=B+F$, then we must have $|(x)| < R_1 \leqslant 1$, $|y| < R_2 \leqslant 1$

Here as well as in what follows (a) stands for the set of A parameters $a_1, a_2, \ldots, a_A$ with similar notations for (b), (c), (d), (e) and (f).

The point $x = 0, y = 0$ is a singular point of the partial differential equations satisfied by $S(x, y)$. The behaviour of $S(x, y)$ in the neighbourhood of $x = 0$, $y=0$ is given by [9, (36), pp. 23-24]

$$S(x, y) = 0\,(\,|\,x\,|^{d_{h_1}}\;\;|\,y\,|^{f_{h_2}}\,)$$

where $h_1 = 1, 2, \ldots, r\,; = 1, 2, \ldots, l$

Similarly the behaviour of the associated function $S_1(x, y)$ (which corresponds to the case $p = 0$) at infinity is given by [9, (40), p. 27]

$$S_1(x, y) = 0\,(\,|\,x\,|^{c_{j_1}-1}\;\;|\,y\,|^{e_{j_2}-1})$$

where $j_1 = 1, 2, \ldots, q\,; j_1 = 1, 2, \ldots, k.$

In the present paper we give some finite as well as infinite integrals involving the generalized function of two variables and discuss their numerous particular cases.

2. FINITE INTEGRALS

The first integral to be proved is

$$(2{\cdot}1)\quad \int_0^1 u^{\lambda-1}(1-u^2)^{-\frac{1}{2}\mu} P_\nu^{\mu}(u)\, S\left[\begin{array}{l|l|l} \begin{bmatrix} p\,, & 0 \\ A-p\,; & B \end{bmatrix} & (a)\,;\,(b) & \\ \begin{pmatrix} q\,, & r \\ C-q\,; & D-r \end{pmatrix} & (c)\,;\,(d) & xu^{2n}, yu^{2n} \\ \begin{pmatrix} k\,, & l \\ E-k\,; & F-l \end{pmatrix} & (e)\,;\,(f) & \end{array}\right] du$$

$$= (2n)^{\mu-1} S\left[\begin{array}{l|c|c} \begin{bmatrix} p+2n\,, & 0 \\ A-p\,, & B+2n \end{bmatrix} & \begin{array}{r}\triangle(2n, \lambda), (a)\,;\,\triangle\,\{n, \frac{1}{2}(1+\lambda-\mu-\nu)\}, \\ \triangle\{n, \frac{1}{2}(2+\lambda-\mu+\nu)\}, (b)\end{array} & \\ \begin{pmatrix} q\,, & r \\ C-q\,, & D-r \end{pmatrix} & (c)\,;\,(d) & x, y \\ \begin{pmatrix} k\,, & l \\ E-k\,, & F-l \end{pmatrix} & (e)\,;\,(f) & \end{array}\right]$$

where $\triangle\,(n, a)$ stands for the set of n parameters $\dfrac{a}{n}, \dfrac{a+1}{n}, \ldots, \dfrac{a+n-1}{n}$. The above formula is valid under the following alternative sets of conditions

(i) $2(p+q+r) > A+B+C+D$, $|\arg.(x)| < (p+q+r-\frac{1}{2}A-\frac{1}{2}B-\frac{1}{2}C-\frac{1}{2}D)\,\pi$,
$2(p+k+l) > A+B+E+F$, $|\arg.(y)| < (p+k+l-\frac{1}{2}A-\frac{1}{2}B-\frac{1}{2}E-\frac{1}{2}F)\,\pi$,
$Re.(\lambda+2nd_{h_1}+2nf_{h_2}) > 0$ $(h_1 = 1, 2, \ldots, r\,; h = 1, 2, \ldots, l)$

(ii) $A+C<B+D$, $A+E<B+F$ $(A+C=B+D$, $A+E=B+F$ then $|\,x\,|, |\,y\,| < 1)$
$Re.(\lambda+2nd_{h_1}+2nf_{h_2}) > 0$ $(h_1 = 1, 2, \ldots, r\,; h_2 = 1, 2, \ldots, l)$
$Re.(\mu) < 1$

To prove this formula substitute the double integral in place of $S(x, y)$ occurring in the integrand of (2·1), interchange the order of integration and then evaluate the inner integral with the help of[1]

$$\int_0^1 u^{\lambda-1} (1-u^2)^{-\frac{1}{2}\mu} P_\nu^{\mu}(u)\, du = \frac{\pi^{\frac{1}{2}} 2^{\mu-\lambda} \Gamma(\lambda)}{\Gamma\{\frac{1}{2}(1+\lambda-\mu-\nu)\} \Gamma\{\frac{1}{2}(2+\lambda-\mu+\nu)\}}$$

$$Re.(\lambda) > 0, Re.(\mu) < 1 ;$$

and make use of the Gauss's multiplication formula [3, p. 4]

$$\Gamma(mz) = (2\pi)^{\frac{1}{2}-\frac{1}{2}m} m^{mz-\frac{1}{2}} \prod_{k=0}^{m-1} \Gamma(z+k/m)$$

Regarding the interchange of the order of integration, it is observed that the u-integral is absolutely convergent if $Re.(\lambda+2ns+2nt) > 0$, $Re.(\mu) < 1$, the double contour integral converges absolutely under the conditions referred to earlier, and the convergence of the repeated integral follows from that of the integral in (2·1). Hence the interchange of the order of integration is justified.

Particular Cases. (*i*) In (2·1) on taking $A = B = 0$ we get

$$(2·2) \qquad \int_0^1 u^{\lambda-1} (1-u^2)^{-\frac{1}{2}\mu} P_\nu^{\mu}(u)\, G_{C,D}^{r,q}\left(xu^{2n} \middle| \begin{matrix}(c)\\(d)\end{matrix}\right) G_{E,F}^{l,k}\left(yu^{2n} \middle| \begin{matrix}(e)\\(f)\end{matrix}\right) du$$

$$= (2n)^{\mu-1} S \left[\begin{matrix} \begin{bmatrix} 2n, & o \\ o, & 2n \end{bmatrix} \\ \begin{pmatrix} q, & r \\ C-q, & D-r \end{pmatrix} \\ \begin{pmatrix} k, & l \\ E-k, & F-l \end{pmatrix} \end{matrix} \middle| \begin{matrix} \triangle(2n, \lambda); \triangle\{n, \frac{1}{2}(1+\lambda-\mu-\nu)\}, \\ \triangle\{n, \frac{1}{2}(2+\lambda-\mu+\nu)\} \\ (c); (d) \\ (e); (f) \end{matrix} \middle| x, y \right]$$

valid under the conditions

(*i*) $2(q+r) > C+D$, $|\arg.(x)| < (q+r-\frac{1}{2}C-\frac{1}{2}D)\pi$,
$2(k+l) > E+F$, $|\arg.(y)| < (k+l-\frac{1}{2}E-\frac{1}{2}F)\pi$,
$Re.(\lambda+2nd_{h_1}+2nf_{h_2}) > 0$ $(h_1 = 1, 2, \ldots, r;\ h_2 = 1, 2, \ldots, l)$
$Re.(\mu) < 1$

(*ii*) $C < D$, $E < F$, (If $C = D$, $E = F$, we must have $|x| < 1$, $|y| < 1$)
$Re.(\lambda+2nd_{h_1}+2nf_{h_2}) > 0$, $Re.(\mu) < 1$.

(*ii*) Next we take $A = p$, $E = k$, $l = 1$, $f_1 = 0$ and make $y \to 0$, then on replacing $A+C$ by A, $B+D$ by B, $A+q$ by S along with the necessary changes in the parameters we get the following known integral [8, pp. 226]

$$(2·3) \qquad \int_0^1 u^{\lambda-1} (1-u^2)^{-\frac{1}{2}\mu} P_\nu^{\mu}(u)\, G_{A,B}^{r,s}\left(xu^{2n} \middle| \begin{matrix}(a)\\(b)\end{matrix}\right) du$$

$$= (2n)^{\mu-1} G_{A+2n, B+2n}^{r, s+2n}\left(x \middle| \begin{matrix} \nabla(2n, \lambda), (a) \\ (b), \nabla\{n, \frac{1}{2}(1+\lambda-\mu-\nu)\}, \nabla\{n, \frac{1}{2}(2+\lambda-\mu+\nu)\} \end{matrix}\right)$$

where $\nabla(r, a)$ stands for a set of r parameters

$$1-\frac{a}{r},\ 1-\frac{a+1}{r},\ \ldots,\ 1-\frac{a+r-1}{r}$$

and
$$2(r+s) > A+B,\ |\arg.(x)| < (r+s-\tfrac{1}{2}A-\tfrac{1}{2}B),$$
$$Re.(\lambda+2nb_j) > 0,\ (j=1,2,\ldots,r)$$
$$Re.(\mu) < 1$$
In case $A=B$, $|x| < 1$ and $|y| < 1$

An interesting special case of (2·3) is the known formula [8, p. 228]

$$\int_0^1 x^{p-1}(1-x^2)^{-\frac{1}{2}\sigma} J_\lambda\left(\frac{x}{a}\right) J_\mu\left(\frac{x}{a}\right) P_\nu^\sigma(x)\,dx$$

$$= \frac{2^{\sigma-1}\,\Gamma(\frac{1}{2}\delta)\,\Gamma(\frac{1}{2}\delta+\frac{1}{2})}{a^{\lambda+\mu}\,\Gamma(\lambda+1)\,\Gamma(\mu+1)\,\Gamma\{\frac{1}{2}(\delta-\nu-\sigma+1)\}\,\{\frac{1}{2}(\delta+\nu-\sigma+2)\}}$$

$$\times\ {}_4F_5\left[\begin{matrix}\frac{1}{2}\delta,\ \frac{1}{2}\delta+\frac{1}{2},\ \frac{1}{2}(\lambda+\mu+1),\ \frac{1}{2}(\lambda+\mu+2)\ ; \\ \frac{1}{2}(\delta-\nu-\sigma+1),\ \frac{1}{2}(\delta+\nu-\sigma+2),\ \lambda+1,\ \mu+1,\ \lambda+\mu+1\ ;\end{matrix}\ -\frac{1}{a^2}\right]$$

where $\delta=(\lambda+\mu+p)$, $Re(\delta)>0$, $Re(\sigma)<1$ (*see* also [11], pp. 420-421, formula (2·1) with $C=1$) which follows when we make use of the relation [3, p. 218]

$$G_{2,4}^{1,2}\left(t^2\,\middle|\,\begin{matrix}a+\frac{1}{2},\ a\\ b+a,\ a-c,\ a+c,\ a-b\end{matrix}\right)=\pi^{\frac{1}{2}}\,t^{2a}J_{b+c}(t)\,J_{b-c}(t).$$

(*iii*) If we set $p=A=m$, $B=n$, $q=k=C=E=l$, $r=l=1$, $D=F=p+1$ $d_1\cdot f_1=0$ and replace b_j, $1-c_j$, $1-d_j$, $1-e_j$ and $1-f_j$ by c_j, b_j, d_j, b'_j and d'_j respectively, the generalized function $S(x, y)$ reduces to Kampé de Fériet's double hypergeometric function. Thus (2·1) gives us

$$(2\cdot4)\qquad \int_0^1 u^{\lambda-1}(1-u^2)^{-\frac{1}{2}\mu}P_\nu^\mu(u)\,.\,F\left(\begin{matrix}m\\ l\\ \\ n\\ p\end{matrix}\left|\begin{matrix}a_1,\ldots,a_m\\ b_1,b'_1;\ldots;b_e,b'_e\\ \\ c_1,c_2,\ldots,c_n\\ d_1,d'_1;\ldots;d_p,d'_p\end{matrix}\right|\,xu^{2r},yu^{2r}\right)du$$

$$=(2r)^{\mu-1}\frac{\prod_{k=1}^{2r}\Gamma\left(\frac{\lambda+k}{2r}\right)}{\prod_{k=1}^{r}\Gamma\left\{\frac{1}{2r}(\lambda+1-\mu-\nu+2kr)\right\}\prod_{k=1}^{r}\Gamma\left\{\frac{1}{2r}(\lambda+2-\mu+\nu)\right\}}$$

$$\times\ F\left(\begin{matrix}m+2r\\ l\\ \\ n+2r\\ p\end{matrix}\left|\begin{matrix}a_1,a_2,\ldots,a_m,\Delta(2r,\lambda)\\ b_1,b'_1;\ldots;b_e,b'_e\\ \\ c_1,c_2,\ldots,c_n,\Delta\{r,\frac{1}{2}(\lambda+1-\mu-\nu)\},\Delta\{r,\frac{1}{2}(2+\lambda-\mu+\nu)\}\\ d_1,d'_1;\ldots;d_p,d'_p\end{matrix}\right|\,x,y\right)$$

valid if $(m+l) < (n+p+1)$, $\{m+l=n+p+1$, then $|x|, |y| < 1\}$

$Re.\ (\lambda) > 0,\ Re(\mu) < 1$

or if $m+l+1 > n+p$, then

$|\arg.\ (y)|, |\arg.\ (x)| < (m+l-n-p)\ \pi/2,$

and $Re(\lambda) > 0,\ Re(\mu) < 1.$

The second finite integral to be established is

$$(2\cdot5)\qquad \int_0^1 u^{\sigma}(1-u)^{\beta} P_n^{(\alpha,\beta)}(1-2u)\ S\left[\begin{array}{l|l|l} \left[\begin{matrix} p, & o \\ A-p, & B \end{matrix}\right] & (a)\ ;\ (b) & \\ \left(\begin{matrix} q, & r \\ C-q, & D-r \end{matrix}\right) & (c)\ ;\ (d) & xu^m, yu^m \\ \left(\begin{matrix} k, & l \\ E-k, & F-l \end{matrix}\right) & (e)\ ;(f) & \end{array}\right]$$

$$= \frac{\Gamma(\beta+n+1)\,(-1)^n}{\lfloor n \ m^{\beta+1}}$$

$$\times S\left[\begin{array}{l|l|l} \left[\begin{matrix} p+2m, & o \\ A-p & , B+2m \end{matrix}\right] & \begin{matrix}\Delta(m, \sigma+1), \Delta(m, 1-\alpha+\sigma), (a)\ ;\ \Delta(m, \beta+\sigma+u+2), \\ \Delta(m, 1-\alpha+\sigma+n), (b)\end{matrix} & \\ \left(\begin{matrix} q & , & r \\ C-q & ; & D-r \end{matrix}\right) & (c)\ ;\ (d) & x, y \\ \left(\begin{matrix} k & , & l \\ E-k & ; & F\ l \end{matrix}\right) & (e)\ ;\ (f) & \end{array}\right]$$

valid under the following sets of conditions

(i) $2(p+q+r) > A+B+C+D$, $|\arg.\ (x)| < (p+q+r-\frac{1}{2}A-\frac{1}{2}B-\frac{1}{2}C-\frac{1}{2}D, \pi,$
$2(p+k+l) > A+B+E+F$, $|\arg.\ (y)| < (p+k+l-\frac{1}{2}A-\frac{1}{2}B-\frac{1}{2}C-\frac{1}{2}D)\ \pi,$
$Re.\ (\sigma+md_{h_1}+mf_{h_2}) > -1\ \{h_1 = 1, 2, \ldots, r\ ;\ h_2 = 1, 2, \ldots, l\},$
$Re.\ (\beta) > -1.$

(ii) $A+C < B+D$, $A+E < B+F$ {for $A+C = B+D$, $A+E = B+F$; $|x|, |y| < 1\}$
$Re.\ (\sigma+md_{h_1}+mf_{h_2}) > -1$, $Re.\ (\beta) > -1$.

The proof of this formula is parallel to that of (2·1), and the final result is obtained by using the known integral [4, p. 284]

$$\int_0^1 u^{\sigma}(1-u)^{\beta} P_n^{(\alpha,\beta)}(1-2u)\,du = \frac{\Gamma(\sigma+1)\,\Gamma(\beta+n+1)\,\Gamma(\alpha-\sigma+n)}{\lfloor n\ \Gamma(\alpha-\sigma)\ \Gamma(\beta+\sigma+n+2)}$$

where $Re.\ (\sigma) > -1$, and $Re.\ (\beta) > -1$.

Particular cases. (i) Put $A = B = p = 0$ and use the result [9, p. 40].

We shall get

$$(2\cdot6)\qquad \int_0^1 u^\sigma (1-u)^\beta P_n^{(\alpha,\beta)}(1-2u)\, G_{C,D}^{r,q}\left(xu^m \,\middle|\, \begin{matrix}(c)\\(d)\end{matrix}\right) G_{E,F}^{l,k}\left(yu^m \,\middle|\, \begin{matrix}(e)\\(f)\end{matrix}\right) du$$

$$= \frac{\Gamma(\beta+n+1)\,(-1)^n}{\lfloor n \;\; m^{\beta+1}}$$

$$\times S\left[\begin{matrix}\begin{bmatrix} 2m\,, & 0\\ 0\,, & 2m\end{bmatrix}\\ \begin{pmatrix} q\,, & r\\ C\text{-}q\,, & D\text{-}r\end{pmatrix}\\ \begin{pmatrix} k\,, & l\\ E\text{-}k\,, & F\text{-}l\end{pmatrix}\end{matrix}\,\middle|\, \begin{matrix}\Delta(m,\sigma+1),\ \Delta(m,1-\alpha+\sigma)\ ;\ \Delta(m,\beta+\sigma+n+2),\\ \Delta(m,1-\alpha+\sigma-n)\\ (c)\ ;\ (d)\\ (e)\ ;\ (f)\end{matrix}\,\middle|\, x, y\right]$$

The above result is valid under the following alternative sets of conditions

(i) $2(q+r) > C+D$, $|\arg.(x)| < (q+r-\tfrac{1}{2}C-\tfrac{1}{2}D)\pi$,
$2(l+k) > E+F$, $|\arg.(y)| < (l+k-\tfrac{1}{2}E-\tfrac{1}{2}F)\pi$,
$Re.(\sigma+md_{h_1}+mf_{h_2}) > -1$, $\{h_1 = 1, 2, \ldots, r\,;\ h_2 = 1, 2, \ldots, l\}$
$Re.(\beta) > -1$.

(ii) $C < D$, $E < F$ (or $C = D$, $E = F$, $|x| < 1$, $|y| < 1$)
$Re.(\sigma+md_{h_1}+mf_{h_2}) > 1$, $\{h_1=1, 2, \ldots, r\,;\ h_2 = 1, 2, \ldots, l\}$
$Re.(\beta) > -1$.

(iii) If we take $A = p$, $E = k$, $l = 1$, $f_1 = 0$ and replace $A + C$ by A, $B+D$ by B, $A + q$ by S together with the appropriate changes in the parameters and then make $y \to 0$. On using the relations [9, (50), p. 39] and [3, (1), p. 215], we then obtain

$$(2\cdot7)\qquad \int_0^1 u^\sigma (1-u)^\beta P_n^{(\alpha,\beta)}(1-2u)\, G_{A,B}^{r,s}\left(xu^m \,\middle|\, \begin{matrix}(a)\\(b)\end{matrix}\right) du$$

$$= \frac{\Gamma(\beta+n+1)\,(-1)^n}{\lfloor n \;\; m^{\beta+1}}\, G_{A+2m,\,B+2m}^{r,\,s+2m}\left(x\,\middle|\, \begin{matrix}\nabla(m,\sigma+1),\ \nabla(m,1-\alpha+\sigma),\ (a)\\ (b),\ \nabla(m,\beta+\sigma+n+2),\ \nabla\, m, 1\ \alpha+\sigma-n)\end{matrix}\right)$$

where, as before, $\nabla(n, \alpha)$ stands for a set of n parameters

$$1-\frac{\alpha}{n},\ 1-\frac{\alpha+1}{n},\ \ldots,\ 1-\frac{\alpha+n-1}{n}$$

The factors $\prod_{k=0}^{m-1}\Gamma\left(\frac{1-\alpha+\sigma+k}{m}+S\right)$ and $\prod_{k=0}^{m-1}\Gamma\left(\frac{1-\alpha+\sigma-n+k}{m}+S\right)$ present in the numerator and denominator respectively of the contour integral representation of the G-function [3, p. 207] on the right hand side of (2·7) may conveniently

be shifted to the denominator and the numerator respectively by means of the formula [3, p. 3]

$$\Gamma(z)\,\Gamma(1-z)=\frac{\pi}{\sin \pi z}$$

so as to put (2·7) in the known form [7, p. 198]

$$(2\cdot8)\qquad \int_0^1 u^{\sigma}(1-u)^{\beta} P_n^{(\alpha,\beta)}(1-2u)\, G_{A,\,B}^{r,\,s}\left(xu^m \,\middle|\, \begin{matrix}(a)\\(b)\end{matrix}\right) du$$

$$=\frac{\Gamma(\beta+n+1)}{\lfloor n \; m^{\beta+1}}\, G_{A+2m,\,B+2m}^{r+m,\,s+m}\left(x \,\middle|\, \begin{matrix}\nabla\,(m,\sigma+1),(a),\,\Delta\,(m,\,\alpha-\sigma\\ \Delta\,(m,\,\alpha-\sigma+n),(b),\,\nabla\,\{m,\,\beta+\sigma+n+2\}\end{matrix}\right)$$

This result is valid under the following alternative sets of conditions :—

(i) $2(r+s)>A+B$, $|\arg.(x)|<(r+s-\tfrac{1}{2}A-\tfrac{1}{2}B)\pi$

$Re.\,(\sigma+mb_{h_1})>-1 \qquad (h_1=1,2,\ldots,r)$

$Re.\,(\beta)>-1$.

(ii) $A<B$ (or $A=B$, $|x|<1$)

$Re.\,(\sigma+mb_{h_1})>-1, \qquad (h_1=1,2,\ldots,r)$

$Re.\,(\beta)>-1$.

When $m=1$, $b_1=0$, $r=1$, $s=A$ and B replaced by $B+1$, the formula (2·8) reduces to an interesting integral given recently by Bhonsle [2, p. 188]

3. AN INFINITE INTEGRAL

In this section we give an infinite integral involving the generalized associated function $S_1(x,y)$ of two variables. The formula to be proved is

$$(3\cdot1)\qquad \int_0^\infty u^{\lambda-1}(1+u)^{-\frac{1}{2}}\,[u^{\frac{1}{2}}+(1+u)^{\frac{1}{2}}]^{2\mu}.\, S_1\left[\begin{matrix}\begin{bmatrix}0 & , & 0\\ A & , & B\end{bmatrix}\\ \begin{pmatrix}q & , & r\\ C-q & , & D-r\end{pmatrix}\\ \begin{pmatrix}k & , & l\\ E-k & , & F-l\end{pmatrix}\end{matrix}\,\middle|\,\begin{matrix}(a)\,;\,(b)\\ \\(c)\,;\,(d)\\ \\(e)\,;(f)\end{matrix}\,\middle|\, xT,yT\right] du$$

$$=\frac{\Gamma(\frac{1}{2}-\lambda-\mu)\,\Gamma(\frac{1}{2}+\lambda+\mu)\,m^{2\lambda-\frac{1}{2}}}{\sqrt{\pi}\,(m+n)^{\lambda+\mu}\,(m-n)^{\lambda-\mu}}$$

$$\times\; S\left[\begin{matrix}\begin{bmatrix}2m & , & 0\\ -A & , & B+2m\end{bmatrix}\\ \begin{pmatrix}q & , & r\\ C-q & , & D-r\end{pmatrix}\\ \begin{pmatrix}k & , & l\\ E-k & , & F-l\end{pmatrix}\end{matrix}\,\middle|\,\begin{matrix}\Delta(2m,2\lambda),(a)\,;\,\Delta\,(m+n,\frac{1}{2}+\lambda+\mu),\\ \Delta\,(m-n,\frac{1}{2}+\lambda-\mu),(b)\\(c)\,;\,(d)\\ \\(e)\,;\,(f)\end{matrix}\,\middle|\,\delta x,\,\delta y\right]$$

where $\delta = \frac{(-1)^{m+n}}{(m+n)^{m+n}\,(m-n)^{m-n}}$, m and n are positive and non-negative integers respectively such that $m > n$ and $T = u^m \{u^{\frac{1}{2}}+(1+u)^{\frac{1}{2}}\}^{2n}$

The above formula is valid under the following conditions :—

$$2(q+r) > A+B+C+D, \; |\arg. (x)| < (q+r-\tfrac{1}{2}A-\tfrac{1}{2}B-\tfrac{1}{2}C-\tfrac{1}{2}D)\,\pi,$$

$$2(k+l) > A+B+E+F, \; |\arg. (y)| < (k+l-\tfrac{1}{2}A-\tfrac{1}{2}B-\tfrac{1}{2}E-\tfrac{1}{2}F)\,\pi,$$

$$Re.\,(\lambda+md_{h_1}+mf_{h_2}) > 0, \; (h_1 = 1, 2, \ldots, r\,;\; h_2 = 1, 2, \ldots, l)$$

$$Re.\{\tfrac{1}{2}-\lambda-\mu-(m+n)\,(c_{j_1}+e_{j_2}-2)\} > 0, \; (j_1 = 1, 2, \ldots, q\,;\; j_2 = 1, 2, \ldots, k).$$

To prove (3·1) we substitute the value of the function $S_1(x, y)$ in the integrand of (3·1), interchange the order of integration, which is justified in the light of the conditions mentioned above, then evaluate the inner integral with the help of (4, p. 311].

$$\int_0^\infty u^{\lambda-1}\,(1+u)^{-\frac{1}{2}}\,[u^{\frac{1}{2}}+(1+u)]^{2\mu}\,du = \frac{2^{1-2\lambda}\,\Gamma(2\lambda)\,\Gamma(\frac{1}{2}-\lambda-\mu)}{\Gamma(\frac{1}{2}+\lambda-\mu)}.$$

where $Re\,(\lambda) > 0$ and $Re(\frac{1}{2}-\lambda-\mu) > 0$; and

(3·1) follows immediately.

Particular Cases : (*i*) If we take $A = 0$, $B = 0$ and use the relation [9 (3) p. 42] we get

$$(3{\cdot}2) \qquad \int_0^\infty u^{\lambda-1}\,(1+u)^{-\frac{1}{2}}\,[u^{\frac{1}{2}}+(1+u)^{\frac{1}{2}}]^{2\mu}\, G^{r,q}_{C,D}\left(xT\,\middle|\,\begin{matrix}(c)\\(d)\end{matrix}\right) G^{l,k}_{E,F}\left(yT\,\middle|\,\begin{matrix}(e)\\(f)\end{matrix}\right)du$$

$$= \frac{\Gamma(\frac{1}{2}-\lambda-\mu)\,\Gamma(\frac{1}{2}+\lambda+\mu)\,m^{2\lambda-\frac{1}{2}}}{\sqrt{\pi}\,(m+n)^{\lambda+\mu}\,(m-n)^{\lambda-\mu}}$$

$$S\left[\begin{matrix}\begin{bmatrix}2m\,, & o\\ o\,, & 2m\end{bmatrix}\\ \begin{pmatrix}q\,, & r\\ C-q\,, & D-r\end{pmatrix}\\ \begin{pmatrix}k\,, & l\\ E-k\,, & F-l\end{pmatrix}\end{matrix}\;\middle|\;\begin{matrix}\Delta(2m, 2\lambda)\,;\,\Delta(m+n, \frac{1}{2}+\lambda+\mu),\,\Delta(m-n, \frac{1}{2}+\lambda-\mu)\\ (c)\,;\,(d)\\ (e)\,;\,(f)\end{matrix}\;\middle|\;\delta x,\,\delta y\right]$$

Valid under the following conditions

$$2(q+r) > C+D\,; \; |\arg. (x)| < (q+r-\tfrac{1}{2}C-\tfrac{1}{2}D)\,\pi,$$

$$2(l+k) > E+F\,; \; |\arg. (y)| < (k+l-\tfrac{1}{2}E-\tfrac{1}{2}F)\,\pi;$$

$$Re.\,(\lambda+md_{h_1}+mf_{h_2}) > 0, \; (h_1 = 1, 2, \ldots, r\,;\; h_2 = 1, 2, \ldots, l)$$

$$Re.\,\{\tfrac{1}{2}-\lambda-\mu-(m+n)\,(c_{j_1}+e_{j_2}-2)\} > 0, \; (j_1=1, 2, \ldots, q\,;\; j_2 = 1, 2, \ldots, k)$$

(*ii*) On the other hand if we set $A=0$, $B=n$, $q = k = C = E = l$, $r = l = 1$, $D = F = p+1$, $d_1 = f_1 = 0$ and replace b_j, $1-c_j$, $1-d_j$, $1-e_j$ and $1-f_j$ by c_j, b_j, d_j, b'_j, d'_j respectively, on using the relation [9 (1) pp. 40-41] we get

$$(3\cdot3)\quad \int_0^\infty u^{\lambda-1}(1+u)^{-\frac{1}{2}}[u^{\frac{1}{2}}+(1+u)^{\frac{1}{2}}]^{2\mu} F\left(\begin{array}{c|c|c} \begin{array}{c} o \\ l \\ \\ n \\ p \end{array} & \begin{array}{l} \\ b_1, b'_1; \ldots; be, b'e \\ \\ c_1, c_2 \ \ldots, c_n \\ d_1, d'_1; \ldots; d_p, d'_p \end{array} & xT, yT \end{array}\right)$$

$$= \frac{\Gamma(\frac{1}{2}-\lambda-\mu)\,\Gamma(\frac{1}{2}+\lambda+\mu)\, m^{2\lambda-\frac{1}{2}}}{\sqrt{\pi}\,(m+n)^{\lambda+\mu}(m-n)^{\lambda-\mu}}$$

$$\times F\left(\begin{array}{c|l|c} \begin{array}{c} 2m \\ l \\ \\ n+2m \\ p \end{array} & \begin{array}{l} \triangle(2m, 2\lambda) \\ b_1, b'_1; \ldots; b_e, b'_e \\ \\ c_1, c_2, \ldots, c_n, \triangle\{m+n, \frac{1}{2}+\lambda+\mu\}, \triangle\{m\ \ n, \frac{1}{2}-\lambda-\mu\} \\ d_1, d_1; d_p, d'_p \end{array} & \delta x, \delta y \end{array}\right)$$

where $\delta = \dfrac{(-1)^{m+n}}{m+n)^{m+n}(m-n)^{m-n}}$, $m > n$ and $T = u^m\{u^{\frac{1}{2}}+(1+u)^{\frac{1}{2}}\}^{2n}$.

The above formula is valid under the following conditions

(*i*) $l+1 > n+p$,

$|\arg. x| < (l-n-p+1)\pi/2$, $|\arg. y| < (l-n-p+1)$

$Re(\lambda) > 0$

$Re\{\frac{1}{2}-\lambda-\mu+(m+n)(b_j+b'_j)\} > 0 \qquad (j = 1, 2, \ldots l)$

Or

(*ii*) $l < n+p+1$,

$Re(\lambda) > 0$

$Re\{\frac{1}{2}-\lambda-\mu+(m+n)(b_j+b'_j)\} > 0 \qquad (j = 1, 2, \ldots, l)$

(*iii*) Lastly, if we take $A=0$, $E=k$, $l=1$, $f_1=0$, replace y by $-y$, C by A, $B+D$ by B, together with te appropriate changes in the parameters (b), (c), (d), (e), etc., and then make $y \to 0$, use the relations [10, (34) pp. 31–39] and [3, p. 215], and we get

$$(3\cdot4)\qquad \int_0^\infty u^{\lambda-1}(1+u)^{-\frac{1}{2}}[u^{\frac{1}{2}}+(1+u)^{\frac{1}{2}}]^{2\mu}\, G^{r,\,q}_{A,\,B}\left(xT\,\Big|\begin{matrix}(a)\\(b)\end{matrix}\right) du$$

$$= \frac{\Gamma(\frac{1}{2}-\lambda-\mu)\,\Gamma(\frac{1}{2}+\lambda+\mu)\, m^{2\lambda-\frac{1}{2}}}{\sqrt{\pi}(m+n)^{\lambda+\mu}(m+n)^{\lambda-\mu}}$$

$$\times\ G^{r,\,q+2m}_{A+2m,\,B+2m}\left(\delta x\,\Big|\begin{matrix}\nabla(2m, 2\lambda), (a)\\ (b), \nabla\{m+n, \frac{1}{2}+\lambda+\mu\}, \nabla\{m-n, \frac{1}{2}+\lambda-\mu\}\end{matrix}\right)$$

with the usual interpretations for the symbols used.

The factors $\prod\limits_{k=0}^{m+n}\Gamma\left(\dfrac{\frac{1}{2}+\lambda+\mu+k}{m+n}+S\right)$ present in the denominator of the contour integral representation of the G-function on the right hand side of (3·4), when taken to the numerator, enable us to put the formula (3·4) in the known form [6, p. 347].

$$(3\cdot5)\qquad \int_0^\infty u^{\lambda-1}(1+u)^{-\frac{1}{2}}[u^{\frac{1}{2}}+(1+u)^{\frac{1}{2}}]^{2\mu}\, G^{r,\,q}_{A,\,B}\left(xT\,\Big|\begin{matrix}(a)\\(b)\end{matrix}\right)du$$

$$=\frac{(2\pi)^{1-m-n}\,m^{2\lambda-\frac{1}{2}}}{\sqrt{\pi}\,(m+n)^{\lambda+\mu}\,(m-n)^{\lambda-\mu}}$$

$$\times\; G^{r+m+n,\,q+2m}_{A+2m,\,B+2m}\left(\delta' x\,\Big|\begin{matrix}\nabla(2m,\,2\lambda),\ (a)\\ \nabla\{m+n,\,\frac{1}{2}-\lambda-\mu\},\ (b),\ \nabla\{m-n,\,\frac{1}{2}+\lambda\ \mu\}\end{matrix}\right)$$

where $\delta'=\dfrac{m^{2m}}{(m+n)^{m+n}\,(m-n)^{m-n}}$

This formula holds good under the following conditions :

$$2(q+r)>A+B,\ |\arg.(x)|<\left(q+r-\frac{A}{2}-\frac{B}{2}\right)\pi,$$

$$Re(\lambda+b_j)>0,\qquad (j=1,2,\ldots,r)$$

$$Re\{\tfrac{1}{2}-\lambda-\mu-(m+n)(a_k-1)\}>0\qquad (k=1,2,\ldots,q).$$

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REFERENCES

1. Barnes, E. W. *Quart. J. of Math.* (*Oxford series*) 39 : 97–204, (1908).
2. Bhonsle, B. R. On some results involving Jacobi polynominals, *J. Indian Math. Soc*, 26 : 187–190 (1962).
3. Erdelyi, A. *et al.*, Higher Transcendental functions. Vol I, *McGraw Hill, New York*, (1953).
4. Erdelyi, A. *et al.*, Table of integral transforms, Vol I, *McGraw Hill, New York*, (1954)
5. Erdelyi, A. *et al.*, Tables of integral transforms, Vol. II, *McGraw Hill, New York*, (1954).
6. Saxena, R. K. Some formulae for the *G*-function. *Proc. Cambridge Phil. Soc.*, 59 : 347–350, (1963).
7. Saxena R. K. On some results involving Jacobi polynomials. *J. Indian Math. Soc* 28 : 197-202, (1964).
8. Saran, S. A definite integral involving the *G*-function. *Nieuw Archief voor Wiskunde* 13(3) : 226–229, (1965).
9. Sharma, B. L. A generalized function of two variables. *Thesis approved for the Ph.D. degree of Jodhpur University*, (1964).
10. Sharma, B. L. On the generalized function of two variables, *Annals de Soc. Sci. de Bruxelles*, 79 : 26–10, (1965).
11. Srivastava, H. M. Some integrals involving products of Bessel and Legendre functions, *Rend. Semin. Math. Univ. Podova*, 35 : 418–423, (1965).

ON THE SPIRAL STRUCTURE OF THE GALAXY

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ABSTRACT

The formation of the spiral arm is examined under the influence of the galactic force law, for our Galaxy, given by 'ort and Bottlinger. The condition for the formation of the spiral arm is also deduced from an expression for potential for our Galaxy as derived by Pareaago. The 3kpc-arm is considered to be expanding as matter is being hurled in the galactic plane as a result of instability due to rapid rotation of a central mass.

INTRODUCTION

There is no dearth of theories for the formation of spiral arms. The problem is aggravated not so much because of the complexity of the phenomenon but is due to inadequate theoretical and practical tools. Soundings on the 21-cm line or natural hydrogen[1] has brought about a revolution in galactic structure, and a fairly coherent picture is gradually emerging.

It would not be possible in this paper to give a detail account of the various theories that have been proposed to explain the spiral arm formation. We would give only fleeting references, and other references, though no less important, will be eluded, since it is not the intention here to trace the entire history of the theories given for spiral arm formation. Earlier workers in the domain were Jeans[2], Brown[3], Vogt[4], and Lindblad[5], who tried to explain the spiral structure of the galaxy with varying amount of success. Banerji[6,7] and Lal[8-12] also tackled the problem by considering suitable initial models for the galaxy.

Magnetic forces exist in galaxies, even small magnetic fields, inspite of their small magnitudes produce considerable interaction between the field and the conducting ionized matter, on account of the large linear dimensions over which they operate. Recently a good amount of work has been done by Pacholozyk and Stodolkiewicz[13], Tassoul[14], and Ôki *et al*[20], who have been successful in explaining certain aspects of the spiral arm formation.

According to the Danby[21], 'No single theory will account for all type of spiral structure'. The present paper is devoted in considering the dynamical aspects of the formation of spiral arms. The equations are those of Newtonian particle dynamics. They are used because of thorough investigations into the type of motion that follows from these equations can be a useful prelude to the introduction of more complex forces and less tractable equations.

Case I

The motion of a particle ejected in the plane of the Galaxy under the attraction of a central force will be considered. Initially various interpolation formulae were proposed for the galactic force law. Oort[22], gave a formula of the type.

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$$P(0) = \frac{C_1}{r^2} + C_2 r \tag{1}$$

where $C_1 = 4{\cdot}59 \times 10^{12}$ and $C_2 = 2{\cdot}7 \times 10^{-16}$.
The formula represents the force as the sum of the attraction of the central nucleus assumed to be concentrated in a mass point (first term) and of homogeneous spheroid on an interior point (second term).

The condition for ejection of matter will be

$$w^2_{R1} > C_2 + \frac{C_1}{R_1^3} \tag{2}$$

where $r = R_1$ is the point where ejection commences as a result of instability due to rapid rotation, and w_{R_1} is the angular velocity at that point.

The equation of motion, in the equatorial plane, is

$$\frac{d^2u}{d\theta^2} + u = \frac{P(0)}{h^2u^2} = \frac{1}{h^2}\left(C_1 + \frac{C_2}{u^3}\right) \tag{3}$$

where $h = R_1^2 \omega_{R_1}$, and $u = \frac{1}{r}$. Furthermore, as in usual discussions on the galaxies, we assume that the pressure and viscosity of gas, including turbulent ones, and the magnetic field can be neglected. Multiplying equation(3) by $\frac{2du}{d\theta}$, and integrating

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2}{h^2}\left(C_1 u - \frac{C_2}{u^2}\right) + C \tag{4}$$

when $r = R_1$, the orbit is circular and $\frac{du}{d\theta} = 0$. Hence

$$C = \frac{1}{R_1^2} - \frac{2}{h^2}\left(\frac{C_1}{R_1} - \frac{C_2}{2} R_1^2\right) \tag{5}$$

Assuming that $v < V_e$, where V_e is the velocity of escape, the velocity at any point of the path will be given by

$$v^2 = 2\left(C_1 u - \frac{C_2}{2u^2}\right) + R_1^2 w_{R1}^2 - 2\left(\frac{C_1}{R_1} - \frac{C_2}{2} R_1^2\right) \tag{6}$$

According to Ishida[23], 'all normal galaxies must have experienced enormous explosions of the nucleus 1–10 times in 10^{10} years'. On the above basis he considers the 3 kpc-arm, which is moving towards us with an outward radial motion of 50 km/sec, to be expanding due to an explosion of the galactic nucleus $\sim 2 \times 10^7$ years ago. We, however, consider it to be due to matter being hurled in the galactic plane as a result of rapid rotation of a central mass.

The expression for the finite length of the arm, R_2, will be given by the cubic

$$(R_1C_2)R_2^3 + (2C_1 - R_1^3 C_2 - R_1^3 w_{R1}^2)\, R_2 - 2R_1C_1 = 0 \tag{7}$$

If $$\frac{2C_1}{h^2} = D \quad \text{and} \quad \frac{C_2}{h_2} = E \tag{8}$$

equation (4) in (r, θ) can be written as

$$\left(\frac{dr}{d\theta}\right) = \pm r\sqrt{Cr^2 + Dr - Er^4 - 1} \tag{9}$$

According to the former author[24], 'for a spiral form $\frac{dr}{d\theta}$ must be real, finite, continuous, and of the same sign as r changes with θ. It is therefore necessary that the expression under the root sign must be positive'. The condition for which is

$$Cr^2 + Dr - Er^4 > 1$$

for values of r greater than R_1. When the above condition is satisfied the two values of $\frac{dr}{d\theta}$, which are opposite in sign, but equal in magnitude, show that two spiral arms may emanate from two diametrically opposite points in the galactic disc.

Case II

The motion of a particle ejected in the galactic plane of our Galaxy, under the influence of a central galactic force law, given by Oort as $P(0) = \frac{C_1}{r^2} + C_2 r$, has been discussed. This formula, however, suffers from a slight disadvantage, it is surely not applicable near the galactic centre where it predicts an infinite force. Bottlinger[22] proposed formula of the type

$$P(B_1) = \frac{ar}{1+br^3} \tag{10}$$

where a and b are constants. This formula fulfills the condition that for small r the force should be proportional to r as expected inside a homogeneous sphere or ellipsoid, and that for a large r outside the system the force should approach the inverse square law. It would be interesting to investigate the motion of a particle under the above force law, since it is quite compatible with the general expected trend for our Galaxy. Bottlinger[22] also proposed the formula for the galactic force law, as :

$$P(B_2) = \frac{a_1 r + a_2 r^2}{1+b_1 r^3 + b_2 r^4} \tag{11}$$

where a_1, a_2, b_1, and b_2 are constants, this expression is essentially quite similar to equation (10).

The condition for the ejection of matter will be

$$w^2_{R_1} > \frac{a}{1+bR_1^3} \tag{12}$$

The equation of motion of the particle, when effect due to viscous, magnetic forces, etc. are neglected, is given by

$$\frac{d^2u}{d\theta^2} + u = \frac{P(B_1)}{h^2u^2} = \frac{a}{h^2} \cdot \frac{1}{u^3+b} \tag{13}$$

putting $\quad b = g^3 \quad$ (14)

Integrating equation (13)

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2a}{h^2}\int\frac{du}{u^3+g^3} + C \tag{15}$$

or

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2a}{3g^2h^2}\left[\log(u+g) - \tfrac{1}{2}\log(u^2-gu+g^2) + \frac{3g}{2}\cdot\frac{2}{\sqrt{3g}}\tan^{-1}\frac{(2u-g)}{\sqrt{3g}}\right] + C \tag{16}$$

Now, as before, when $u = \frac{1}{R_1}$ the orbit is circular and $\frac{du}{d\theta} = 0$. Hence,

$$C = \frac{1}{R_1^2} - \frac{2a}{3g^2h^2}\left[\log\frac{1+gR_1}{(1-gR_1+g^2R_1^2)^{\frac{1}{2}}} + \sqrt{3}\tan^{-1}\frac{2-gR_1}{\sqrt{3}gR_1}\right] \tag{17}$$

The velocity at any point of the path will be given by

$$\begin{aligned} v^2 = {} & \frac{2a}{3g^2}\left[\log\frac{u+g}{(u^2-gu+g^2)^{\frac{1}{2}}} + \sqrt{3}\tan^{-1}\frac{2u-g}{\sqrt{3}\,g}\right] + w^2_{R_1}R_1^2 \\ & - \frac{2a}{3g^2}\left[\log\frac{1+gR_1}{(1-gR_1+g^2R_1^2)^{\frac{1}{2}}} + \sqrt{3}\tan^{-1}\frac{2-gR_1}{\sqrt{3}\,gR_1}\right] \end{aligned} \tag{18}$$

For a finite length of the arm let $v = 0$, and there the corresponding value for r be R_2, which will be given by the equation.

$$\begin{aligned} & \log\frac{1+gR_2}{(1-gR_2+g^2R_2^2)^{\frac{1}{2}}} + \sqrt{3}\tan^{-1}\frac{2-gR_2}{\sqrt{3}\,gR_2} + \frac{3g^2\,\omega^2_{R_1}R_1^2}{2a} \\ & - \log\frac{1+gR_1}{(1-gR_1+g^2R_1^2)^{\frac{1}{2}}} - \sqrt{3}\tan^{-1}\frac{2-gR_1}{\sqrt{3}\,gR_1} = 0 \end{aligned} \tag{19}$$

Equation (16) can be written as

$$\left(\frac{dr}{d\theta}\right) = \pm\sqrt{Cr^4 - r^2 + \frac{2ar^4}{3g^2h^2}\left[\log\frac{1+gr}{(1-gr+g^2r^2)^{\frac{1}{2}}} + \sqrt{3}\tan^{-1}\frac{2-gr}{\sqrt{3}\,gr}\right]} \tag{20}$$

As before, it can be argued, that the condition for the ejection of matter, from two diametrically opposite points, will be

$$3g^2R_1^2w^2_{R_1}\,r^2 - 3g^2R_1^4\,w^2_{R_1} - 2ar^2\log\frac{(1+gR_1)\,(1-gr+g^2r^2)^{\frac{1}{2}}}{(1+gr)\,(1-gR_1+g^2R_1^2)^{\frac{1}{2}}}$$

$$-\,2\sqrt{3}\;ar^2\tan^{-1}\frac{\sqrt{3}\,g(r-R_1)}{2-g(r+R_1)+2g^2rR_1} > 0 \tag{21}$$

for values of r greater than R_1

Case III

Parenago derived, as an approrimation, the potential for the galactic system as follows ;

$$\phi(r,0) = \frac{\phi_c}{1+\chi r^2} \tag{22}$$

where
$$\left.\begin{aligned}\phi_c &= -\tfrac{1}{2}\,R_0^2\frac{(A-B)^4}{AB}\\ &\quad 11{\cdot}3\times 10^{14}\ \text{cm}^2/\text{sec}^2\\ \chi &= -\frac{A}{B}R_0^2 = {\cdot}0288\ \text{kpc}^{-2}\end{aligned}\right\} \tag{23}$$
and

In equation (23) A and B are the Oort's constant of galactic rotation, and R_0 is the distance of the sum from the galactic centre.

Now
$$P(Pa) = \frac{2\,\chi\,\phi_c\,r}{(1+\chi\,r^2)^2} \tag{24}$$

The condition for ejection of matter will be

$$w^2_{R_1} > \frac{2\,\chi\,\phi_c}{(1+\chi\,R_1^2)^2} \tag{25}$$

The equation of motion in the equatorial plane, when effect due to viscous, magnetic forces, etc. are neglected, is given by

$$\frac{d^2u}{d\theta^2} + u = \frac{P(Pa)}{h^2u^2} = \frac{2\,\chi\,\phi_c}{h^2}\cdot\frac{u}{(u^2+\chi)^2} \tag{26}$$

Multiplying by $2\frac{du}{d\theta}$, and integrating equation (26)

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{4\chi\,\phi_c}{h^2}\int\frac{u}{(u^2+\chi)^2}\,du \tag{27}$$

or
$$\left(\frac{du}{d\theta}\right)^2 + u^2 = -\frac{2\chi\,\phi_c}{h^2}\,\frac{1}{u^2+\chi} + D \tag{28}$$

Now when $r = R_1$, the orbit is circular, and $\frac{du}{d\theta} = 0$. Hence

$$D = \frac{1}{R_1^2} + \frac{2\chi\,\phi_c}{h^2} \cdot \frac{R_1^2}{1+\chi R_1^2} \tag{29}$$

The velocity at any point of the path will be given by

$$v^2 = w^2_{R_1} \cdot R_1^2 + 2\chi\,\phi_c \left(\frac{R_1^2}{1+\chi R_1^2} - \frac{r^2}{1+\chi r^2} \right) \tag{30}$$

and the expression for the finite length of the arm, R_2, will be given by

$$w^2_{R_1}\; R_1^2 + \frac{2\chi\,\phi_c\,(R_1^2 - R_2^2)}{(1+\chi R_1^2)\,(1+\chi R_2^2)} = 0 \tag{31}$$

From equation (28)

$$\left(\frac{du}{d\theta}\right)^2 = -\frac{E}{u^2+\chi} + D - u^2 \tag{32}$$

where $E = \frac{2\,\chi\,\phi_c}{h^2}$, E and D are both positive

or

$$\left(\frac{du}{d\theta}\right)^2 = \frac{F+G\,u^2 - u^4}{u^2+\chi} \tag{33}$$

where $F = D\chi - E$ and $G = D - \chi$

Changing equation (33) to polars

$$\left(\frac{dr}{d\theta}\right) = \pm \sqrt{\frac{Fr^4 + Gr^2 - 1}{1+\chi\, r^2}} \tag{34}$$

The condition for the ejection of matter will be

$$Fr^4 + Gr^2 > 1 \tag{35}$$

for values of $r > R_1$

We have determined the condition for the ejection of matter along the equatorial plane. At present, quite unfortunately, we can hardly hope of ascertaining definitely which of the expressions for the galactic force law, the first given by Oort, the second by Bottlinger, and the third force law obtained from the expression for potential as given by Parenago, will be able to explain the spiral arm formation in a better qualitative manner. All the attempts are thwarted because of lack of exhaustive and reliable relationship between distance from the galactic centre, and the angular velocity at distances approaching the galactic centre. Once the impasse has been cleared, the expressions for the galactic force law can then be subjected to the ordeal of explaining the spiral formation. The path of the ejected particle too, can then be determined easily. The need of such a data appears extremely essential, though the task is no less formidable. Better theoretical and practical tools are necessary for a clearer understanding of the galactic phenomenon.

The authors also feel the need of a more accurate expression between the galactic force law and the distance from the galactic centre. It appears that the expression proposed by Oort, Bottlinger and Parenago require revision because of recent and rapid advances in galactic structure and galactic dynamics.

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REFERENCES

1. Oort, J. M. *M N.* **118** : 379, (1958).
2. Jeans, J. *Astronomy and Cosmogony*, Chp. XIII, (1929).
3. Brown, E. W. *Obs.* 277, (1928).
4. Vogt, Astron Nachtr, **CCXLII** : 181, 1931 ; **CCXLV** : 281, (1932).
5. Lindblad, B. *Stock. Obs. Ann Band* **12**(4) : (1936).
6. Banerji, A. C. 'Recent advances in Galactic Dynamics', Chap. II, (1942).
7. Banerji, A. C. *Ind. Jour. Phy,* **14**(1) : (1940).
8. Lal, B. B., *Proc. Nat. Acad. Sci.* **12**(2) 108–120, (1942).
9. Lal, B. B. **13**(1) : 28–36, (1943)
10. Lal, B. B. **13**(1) : 19-27, (1943).
11. Lal, B. B. **13**(3) : 165-170, (1943).
12. Lal, B. B. **13** (3) : 179-183, (1943).
13. Pacholczyk, A. G. and Stodolkiewicz, J. S *Bull, de l' Acad. Polonaise des Seiences,* **7** 503, (1958).
14. Tassoul, J. *Annales de l' Observatoire Royale de Belgique, Ser,* **3** : (9), *fasc.* 1. (1962).
15. Wentzel, D. G. *Bull. Ast. Inst. Netherlands,* **15** : 103, (1960).
16. Pismis, P. *Bol. Obs. Tonantzintla Tacubaya,* **3** : 3, (1961).
17. Elvius, A. and Herlofson, N. *Ap. J.* **131** : 304, (1960).
18. Elvius, A. and Lindblad, P. O. *Arkiv f. Astron,* **2**(4) : (1959).
19. Lindblad, P. O. *Stock. Obs. Ann.* **21**(4) : (1960).
20. Oki, T *et al. Supp. Prog. Theo Phy.* **31** : 77–115, (1965).
21. Danby, J. M. A. *A. J.* **70**(7) : 501-512, (1965).
22. Weaver, H F., and Trumpler, R. J. 'Statistical Astronomy' *University of California Press Berkeley and Los Angeles, California,* 548–550.
23. Ishida, K. *Supp. Prog. Theo Phy.,* **31** : 116–130, (1964).
24. Banerji, A. C., *et. al. Phil. Mag. Ser.,* **38** (7) : 118–126, (1939).
25. Lindblad, B., *Handbuch der Physik,* **43** : 21–99, (1959).

ON SOME TRIPLE SERIES EQUATIONS INVOLVING ULTRASPHERICAL POLYNOMIALS

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INTRODUCTION

1. In determining the electrostatic potential for a spherical ring of semi-angles α and β and in the determination of the fundamental wave length for a spherical Helmholtz resonator with two coaxial circular holes Collins[1] was led to the study of triple series relations of the type

$$(1{\cdot}1)\qquad \begin{cases} \sum_{n=0}^{\infty} (2n+1)\, C_n\, P_n(\cos\theta) = 0 & (0 \leqslant \theta < \alpha,\ \beta < \theta \leqslant \pi) \\ \sum_{n=0}^{\infty} (1+H_n)\, C_n\, P_n(\cos\theta) = f(\theta) & (\alpha < \theta < \beta). \end{cases}$$

$$(1{\cdot}2)\qquad \begin{cases} \sum_{n=0}^{\infty} (1+H_n)\, C_n\, P_n(\cos\theta) = f(\theta) & (0 \leqslant \theta < \alpha,\ \beta < \theta \leqslant \pi) \\ \sum_{n=0}^{\infty} (2n+1)\, C_n\, P_n(\cos\theta) = 0 & (\alpha < \theta < \beta). \end{cases}$$

In these equations $P_n(\cos\theta)$ is a Legendre polynomial of degree n, H_n is a known coefficient depending on n, $f(\theta)$ is a given function of the variable θ, and the equations are to be solved for the unknown coefficients C_n. Collins has given a method of solving these equations.

These equations can be regarded as extensions of the dual series equations which have been extensively studied by various workers (for a detailed bibliography *see*[3]). Recently a simple method of determining the coefficients C_n in the dual series equations

$$(1{\cdot}3)\qquad \begin{cases} \sum_{n=0}^{\infty} \dfrac{C_n}{\Gamma(\alpha+n+1)\,\Gamma(\beta+n+3/2)}\, P_n^{(\alpha,\beta)}(\cos\theta) = f_1(\theta), & 0 \leqslant \theta < \phi, \\ \sum_{n=0}^{\infty} \dfrac{C_n}{\Gamma(\beta+n+1)\,\Gamma(\alpha+n+1/2)}\, P_n^{(\alpha,\beta)}(\cos\theta) = f_2(\theta), & \phi < \theta \leqslant \pi, \end{cases}$$

where $P_n^{(\alpha,\beta)}(\cos\theta)$ are Jacobi polynomials, $f_1(\theta)$ and $f_2(\theta)$ are prescribed, has been given by the author[4]. The technique given in[4] is employed here for studying triple series equations involving ultraspherical polynomials which can be regarded as the extentions of the dual series equations (1·3).

We shall mainly be concerned with the triple series relations of the form

(1·4) $$\sum_{n=0}^{\infty} [\Gamma(\lambda+n+1)\,\Gamma(\lambda+n+3/2)]^{-1}\, C_n\, P_n^{(\lambda,\lambda)}(\cos\theta) = l\,(\theta) \qquad 0 \leqslant \theta < \alpha,$$

(1·5) $$\sum_{n=0}^{\infty} [\Gamma(\lambda+n+1)\,\Gamma(\lambda+n+1/2)]^{-1}\, C_n\, P_n^{(\lambda,\lambda)}(\cos\theta) = m\,(\theta) \qquad \alpha < \theta < \beta,$$

(1·6) $$\sum_{n=0}^{\infty} [\Gamma(\lambda+n+1)\,\Gamma(\lambda+n+3/2)]^{-1}\, C_n\, P_n^{(\lambda,\lambda)}(\cos\theta) = n\,(\theta) \qquad \beta < \theta \leqslant \pi,$$

defined as triple series relations of the first kind, and those of the form

(1·7) $$\sum_{n=0}^{\infty} [\Gamma(\lambda+n+1)\,\Gamma(\lambda+n+1/2)]^{-1}\, C_n\, P_n^{(\lambda,\lambda)}(\cos\theta) = l\,(\theta) \quad 0 \leqslant \theta < \alpha,$$

(1·8) $$\sum_{n=0}^{\infty} [\Gamma(\lambda+n+1)\,\Gamma(\lambda+n+3/2)]^{-1}\, C_n\, P_n^{(\lambda,\lambda)}(\cos\theta) = m\,(\theta) \quad \alpha < \theta < \beta,$$

(1·9) $$\sum_{n=0}^{\infty} [\Gamma(\lambda+n+1)\,\Gamma(\lambda+n+1/2)]^{-1}\, C_n\, P_n^{(\lambda,\lambda)}(\cos\theta) = n\,(\theta) \quad \beta<\theta\leqslant\pi,\ \lambda>0,$$

defined as the triple series relations of the second kind. The functions $l\,(\theta)$, $m\,(\theta)$ and $n\,(\theta)$ are prescribed and the equations are to be solved for the unknown coefficients C_n. Subsequent analysis will explain the utility of having the factor $\Gamma(\lambda+n+1)$ in all these equations. The main result of this paper is that the solution of the either set of the triple series relations can be reduced to the solution of a set of Fredholm integral equations of the second kind in one independent variable. The analysis given here is purely formal and no attempt is made to justify the various limiting processes.

2. *Some results involving Jacobi polynomials* :—It is convenient to list in this section some results involving Jacobi polynomials for ready reference. These results are given in[2] and they are valid for $\alpha > -1$, $\beta > -1$. The orthogonal property for the Jacobi polynomials can be expressed by the equation

(2·1) $$\int_0^{\pi} (\sin\theta/2)^{2\alpha} (\cos\theta/2)^{2\beta}\, P_n^{(\alpha,\beta)}(\cos\theta)\, P_m^{(\alpha,\beta)}(\cos\theta)\,\sin\theta\, d\theta$$
$$= \frac{2\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)\,\delta m\,n}{(n)\,!\,(\alpha+\beta+2n+1)\,\Gamma(\alpha+\beta+n+1)}$$

where $\delta m\, n$ is a Kronecker delta.

It can be easily shown that

(2·2) $$\int_0^{u} (\sin\theta/2)^{2\alpha} (\cos\theta - \cos u)^{-1/2}\, P_n^{(\alpha,\beta)}(\cos\theta)\,\sin\theta\, d\theta = (2\pi)^{1/2}\, \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+n+3/2)}$$
$$\times\ (\sin u/2)^{2\alpha+1}\, P_n^{(\alpha+1/2,\ \beta-1/2)}(\cos u)$$

(2·3) $$\int_u^{\pi} (\cos\theta/2)^{2\beta} (\cos u - \cos\theta)^{-1/2}\, P_n^{(\alpha,\beta)}(\cos\theta)\,\sin\theta\, d\theta$$
$$= (2\pi)^{1/2}\, \frac{\Gamma(\beta+n+1)}{\Gamma(\beta+n+3/2)} (\cos u/2)^{2\beta+1}\, P_n^{(\alpha-1/2,\beta+1/2)}(\cos u).$$

Other results which will be used in later work are :

(2·4) $$\frac{d}{d\theta}\{(\sin\theta/2)^{2\alpha} P_n^{(\alpha,\beta)}(\cos\theta)\} = \frac{(n+\alpha)}{2}\sin\theta\,(\sin\theta/2)^{2\alpha-2} P_n^{(\alpha-1,\beta+1)}(\cos\theta).$$

(2·5) $$\frac{d}{d\theta}\{(\cos\theta/2)^{2\beta} P_n^{(\alpha,\beta)}(\cos\theta)\} = -\frac{(n+\beta)}{2}\sin\theta\,(\cos\theta/2)^{2\beta-2} P_n^{(\alpha+1,\beta-1)}(\cos\theta).$$

3. *Variants of Schlomilch's integral equation* :—We shall use the following two forms, which are easily deduced from the well known solutions of Schlomilch's integral equation. They are :

(*i*) If $f(\theta)$ and $f'(\theta)$ are continuous in $\beta \leqslant \theta \leqslant \pi$, the solution of the integral equation

(3·1) $$f(\theta) = \int_\beta^\theta (\cos u - \cos\theta)^{-\frac{1}{2}} g(u)\, du$$

is given by

(3·2) $$g(u) = \frac{1}{\pi}\frac{d}{du}\int_\beta^u (\cos\theta - \cos u)^{-\frac{1}{2}} \sin\theta\, f(\theta)\, d\theta$$

(*ii*) If $f(\theta)$ and $f'(\theta)$ are continuous in $0 \leqslant \theta \leqslant \alpha$, the solution of

(3·3) $$f(\theta) = \int_\theta^\alpha (\cos\theta - \cos u)^{-\frac{1}{2}} g(u)\, du$$

in given by

(3·4) $$g(u) = -\frac{1}{\pi}\frac{d}{du}\int_u^\alpha (\cos u - \cos\theta)^{-\frac{1}{2}} \sin\theta\, f(\theta)\, d\theta.$$

4. *The triple series relations of the first kind* :—By using the relations (2·1), (2·2) and (2·5), (2·3) the equations (1·4) and (1·6) can be written as

(4·1) $$L(\theta) = (2/\pi)^{\frac{1}{2}} (\sin\theta/2)^{1-2\lambda} \int_0^\theta (\cos u - \cos\theta)^{-\frac{1}{2}} \frac{d}{du}\{(\sin u/2)^{2\lambda}\, l(u)\}\, du$$

$$= \sum_{n=0}^{\infty} [\Gamma(\lambda+n+\tfrac{1}{2})\,\Gamma(\lambda+n+\tfrac{3}{2})]^{-1} C_n P_n^{(\lambda-\frac{1}{2},\lambda+\frac{1}{2})}(\cos\theta),\ 0 \leqslant \theta < \alpha,$$

(4·2) $$N(\theta) = -(2/\pi)^{\frac{1}{2}} (\cos\theta/2)^{1-2\lambda} \int_\theta^\pi (\cos\theta - \cos u)^{-\frac{1}{2}} \frac{d}{du}\{(\cos u/2)^{2\lambda} n(u)\}\, du$$

$$= \sum_{n=0}^{\infty} [\Gamma(\lambda+n+\tfrac{1}{2})\,\Gamma(\lambda+n+\tfrac{3}{2})]^{-1} C_n P_n^{(\lambda+\frac{1}{2},\lambda-\frac{1}{2})}(\cos\theta),\ \beta < \theta \leqslant \pi.$$

Let us write

(4·3) $$\sum_{n=0}^{\infty} [\Gamma(\lambda+n+1)\,\Gamma(\lambda+n+\tfrac{1}{2})]^{-1} C_n P_n^{(\lambda,\lambda)}(\cos\theta) = \begin{cases} f(\theta) & 0 \leqslant \theta < \alpha, \\ m(\theta) & \alpha < \theta < \beta, \\ g(\theta) & \beta < \theta \leqslant \pi. \end{cases}$$

where $f(\theta)$ and $g(\theta)$ are unknown functions to be determined. From (4·3) and the orthogonality relation (2·1) we have

$$(4\cdot4)\qquad C_n = \frac{\Gamma(n+\lambda+\frac{1}{2})}{\Gamma(n+\lambda+1)}\, q_n \left[\int_0^\alpha f(u) + \int_\alpha^\beta m(u) + \int_\beta^\pi g(u)\right]$$

$$\times (\sin u/2 \cos u/2)^{2\lambda+1} P_n^{(\lambda,\lambda)}(\cos u)$$

where $\quad q_n = (n)!\,(2\lambda + 2n + 1)\,\Gamma(2\lambda + n + 1).$

If, in equations (4·1) and (4·2), we substitute the coefficients C_n from the relation (4·4) we get, on interchanging the order of summation and integration, the equations

$$(4\cdot5)\qquad L(\theta) = \left[\int_0^\alpha f(u) + \int_\alpha^\beta m(u) + \int_\beta^\pi g(u)\right] J(u,\theta)\, du$$

$$(4\cdot6)\qquad N(\theta) = \left[\int_0^\alpha f(u) + \int_\alpha^\beta m(u) + \int_\beta^\pi g(u)\right] K(u,\theta)\, du$$

where

$$(4\cdot7)\qquad J(u,\theta) = \sum_{n=0}^{\infty} \frac{q_n}{\Gamma(\lambda+n+1)\,\Gamma(\lambda+n+\frac{3}{2})} (\sin u/2 \cos u/2)^{2\lambda+1} P_n^{(\lambda,\lambda)}(\cos u)$$
$$\times P_n^{(\lambda-\frac{1}{2},\, \lambda+\frac{1}{2})}(\cos\theta)$$

$$(4\cdot8)\qquad K(u,\theta) = \sum_{n=0}^{\infty} \frac{q_n}{\Gamma(\lambda+n+1)\,\Gamma(\lambda+n+\frac{3}{2})} (\sin u/2 \cos u/2)^{2\lambda+1} P_n^{(\lambda,\lambda)}(\cos u)$$
$$\times P_n^{(\lambda+\frac{1}{2},\, \lambda-\frac{1}{2})}(\cos\theta).$$

With the help of relations (2·1), (2·2) and (2·3) it can be shown that

$$(4\cdot9)\qquad J(u,\theta) = (2\pi)^{-1} (\cos\theta/2)^{-1-2\lambda} (\cos u/2)^{2\lambda} \sin u (\cos\theta - \cos u)^{-\frac{1}{2}} H(u-\theta),$$

$$(4\cdot10)\qquad K(u,\theta) = (2\pi)^{-1} (\sin\theta/2)^{-1-2\lambda} (\sin u/2)^{2\lambda} \sin u\, (\cos u - \cos\theta)^{-\frac{1}{2}} H(\theta - u),$$

where $H(t)$ is the Heaviside's unit function. The equations (4·5) and (4·6) can be written as

$$(4\cdot11)\qquad P(\theta) = (2\pi)^{\frac{1}{2}} (\cos\theta/2)^{2\lambda+1} L(\theta) - \int_\alpha^\beta (\cos\theta - \cos u)^{-\frac{1}{2}} (\cos u/2)^{2\lambda} \sin u\; m(u)\, du$$

$$= \left[\int_0^\alpha f(u) + \int_\beta^\pi g(u)\right] (\cos\theta - \cos u)^{-\frac{1}{2}} (\cos u/2)^{2\lambda} \sin u\, du, \qquad 0 \leqslant \theta < \alpha,$$

$$(4\cdot12)\qquad Q(\theta) = (2\pi)^{\frac{1}{2}} (\sin\theta/2)^{2\lambda+1} N(\theta) - \int_\alpha^\beta (\cos u - \cos\theta)^{-\frac{1}{2}} (\sin u/2)^{2\lambda} \sin u\; m(u)\, du$$

$$= \left[\int_0^\alpha f(u) + \int_\beta^\theta g(u)\right] (\cos u - \cos\theta)^{-\frac{1}{2}} (\sin u/2)^{2\lambda} \sin u\, du, \quad \beta < \theta \leqslant \pi.$$

Let us assume that

$$(4\cdot13)\qquad F(\theta) = \int_\theta^\alpha f(u)\, (\cos\theta - \cos u)^{-\frac{1}{2}} (\cos u/2)^{2\lambda} \sin u\, du,$$

(4·14) $$G(\theta) = \int_{\beta}^{\theta} g(u)\,(\cos u - \cos\theta)^{-\frac{1}{2}}\,(\sin u/2)^{2\lambda}\sin u\,du.$$

These integral equations are the variants of Schlomilch integral equation and their solutions are given by the relations

(4·15) $$(\cos u/2)^{2\lambda}\sin u\,f(u) = -1/\pi\,\frac{d}{du}\int_{u}^{\alpha}\frac{F(\theta)\sin\theta}{(\cos u-\cos\theta)^{\frac{1}{2}}}\,d\theta,$$

(4·16) $$(\sin u/2)^{2\lambda}\sin u\,g(u) = 1/\pi\,\frac{d}{du}\int_{\beta}^{u}\frac{G(\theta)\sin\theta}{(\cos\theta-\cos u)^{\frac{1}{2}}}\,d\theta.$$

Hence (4·15) and (4·16) can be written as

(4·17) $$P(\theta)=F(\theta)+1/\pi\int_{\beta}^{\pi}\frac{(\cos u/2)^{2\lambda}}{(\cos\theta-\cos u)^{\frac{1}{2}}}\left(\frac{d}{du}\int_{\beta}^{u}(\cos v-\cos u)^{\frac{1}{2}}\sin v\,G(v)\,dv\right)du$$

$$=F(\theta)-1/\pi\int_{\beta}^{\pi}\frac{d}{du}\left\{\frac{(\cot u/2)^{2\lambda}}{(\cos\theta-\cos u)^{\frac{1}{2}}}\right\}\left(\int_{\beta}^{u}(\cos v-\cos u)^{-\frac{1}{2}}\sin v\,G(v)dv\right)du$$

$$=F(\theta)-\int_{\beta}^{\pi}G(v)\,R(v,\theta)\,dv \qquad 0\leqslant\theta<\alpha,$$

(4·18) $$Q(\theta)=G(\theta)-1/\pi\int_{0}^{\alpha}\frac{(\tan u/2)^{2\lambda}}{(\cos u-\cos\theta)^{\frac{1}{2}}}\left(\frac{d}{du}\int_{u}^{\alpha}(\cos u-\cos v)^{-\frac{1}{2}}\sin v\,F(v)\,dv\right)du$$

$$=G(\theta)+\int_{0}^{\alpha}F(v)\,S(v,\theta)\,dv \qquad \beta<\theta\leqslant\pi,$$

where

(4·19) $$R(v,\theta)=\frac{\sin v}{\pi}\int_{v}^{\pi}(\cos v-\cos u)^{-\frac{1}{2}}\,\frac{d}{du}\left\{\frac{(\cot u/2)^{2\lambda}}{(\cos\theta-\cos u)^{\frac{1}{2}}}\right\}du$$

(4·20) $$S(v,\theta)=\frac{\sin v}{\pi}\int_{0}^{v}(\cos u-\cos v)^{-\frac{1}{2}}\,\frac{d}{du}\left\{\frac{(\tan u/2)^{2\lambda}}{(\cos u-\cos\theta)^{\frac{1}{2}}}\right\}du$$

We can now eliminate either $F(\theta)$ between the equations (4·17) and (4·18) to obtain a Fredholm integral equation of second kind for $G(\theta)$, or alternatively eliminate $G(\theta$ between these equations to obtain a Fredholm integral equation for $F(\theta)$. If we eliminate $F(\theta)$, we obtain

(4·21) $$\Upsilon(\theta) = G(\theta)+\int_{\beta}^{\pi}G(x)\,T(x,\theta)\,dx \qquad \beta<\theta\,\pi\leqslant\pi,$$

where

(4·22) $$\Upsilon(\theta) = Q(\theta)-1/\pi\int_{0}^{\alpha}P(v)\,S(v,\theta)\,dv$$

(4·23) $$T(x,\theta) = 1/\pi^{2}\int_{0}^{\alpha}R(x,v)\,S(v,\theta)\,dv,$$

Alternatively, if we eliminate $G(\theta)$, we obtain

$$(4{\cdot}24)\qquad Z(\theta) = F(\theta) - \int_0^{\alpha} F(x)\, U(x, \theta)\, dx \qquad 0 \leqslant \theta < \alpha,$$

where

$$(4{\cdot}25)\qquad Z(\theta) = P(\theta) + 1/\pi \int_{\beta}^{\pi} Q(v)\, R(v, \theta)\, dv,$$

$$(4{\cdot}26)\qquad U(x, \theta) = 1/\pi^2 \int_{\beta}^{\pi} R(v, \theta)\, S(v, \theta)\, dv.$$

Once $G(\theta)$ or $F(\theta)$ are determined from (4·21) or (4·24) the other follows from (4·17) or (4·18). The functions $f(\theta)$ and $g(\theta)$ are then obtained from (4·15) and (4·16) respectively, the coefficients C_n can be calculated from (4·4).

5. *The triple series relations of the second kind* :—We use (2·2) and (2·3) for rewriting (1·7) and (1·9) in the form

$$(5{\cdot}1)\qquad L_1(\theta) = (2\pi)^{-\frac{1}{2}} (\sin \theta/2)^{-2\lambda-1} \int_0^{\theta} (\cos u - \cos\theta)^{-\frac{1}{2}} (\sin u/2)^{2\lambda}\, l(u) \sin u\, du$$

$$= \sum_{n=0}^{\infty} [\Gamma(\lambda+n+\tfrac{1}{2})\, \Gamma(\lambda+n+\tfrac{3}{2})]^{-1} C_n\, P^{(\lambda+\frac{1}{2},\lambda-\frac{1}{2})} \cos(\theta), \qquad 0 \leqslant \theta < \alpha$$

$$(5{\cdot}2)\qquad N_1(\theta) = (2\pi)^{-\frac{1}{2}} (\cos \theta/2)^{-2\lambda-1} \int_{\theta}^{\pi} (\cos \theta - \cos u)^{-\frac{1}{2}} (\cos u/2)^{2\lambda}\, n(u) \sin u\, du$$

$$= \sum_{n=0}^{\infty} [\Gamma(\lambda+n+\tfrac{1}{2})\, \Gamma(\lambda+n+\tfrac{3}{2})]^{-1} C_n\, P_n^{(\lambda\frac{1}{2},\lambda+\frac{1}{2})} (\cos \theta), \quad \beta < \theta \leqslant \pi.$$

Now we suppose that

$$(5{\cdot}3)\qquad \sum_{n=0}^{\infty} [\Gamma(\lambda+n+1)\, \Gamma(\lambda+n+\tfrac{3}{2})]^{-1} C_n\, P_n^{(\lambda,\lambda)} (\cos \theta) = \begin{cases} f_1(\theta) & 0 \leqslant \theta < \alpha \\ m(\theta) & \alpha < \theta < \beta \\ g_1(\theta) & \beta < \theta \leqslant \pi, \end{cases}$$

where $f_1(\theta)$ and $g_1(\theta)$ are unknown functions to be determined. It is assumed that $f_1(\theta)$ and $g_1(\theta)$ are continuous at $\theta = \alpha$ and $\theta = \beta$, *i.e.* $f_1(\alpha) = m(\alpha)$ and $g_1(\beta) = m(\beta)$. With the help of (2·5) and (2·4) we have

$$(5{\cdot}4)\qquad \sum_{n=0}^{\infty} C_n \frac{\sin \theta (\cos \theta/2)^{2\lambda-2}\, P_n^{(\lambda+1,\ \lambda-1)} (\cos \theta)}{\Gamma(\lambda+n)\, \Gamma(\lambda+n+\frac{3}{2})} = -2 \begin{cases} Dc\,\{f_1(\theta)\} & 0 \leqslant \theta < \alpha \\ Dc\,\{m(\theta)\} & \alpha < \theta < \beta \\ Dc\,\{g_1(\ \ )\} & \beta < \theta \leqslant \pi, \end{cases}$$

$$(5{\cdot}5)\qquad \sum_{n=0}^{\infty} C_n \frac{\sin \theta \sin \theta/2)^{2\lambda-2}\, P_n^{(\lambda-1,\ \lambda+1)} (\cos \theta)}{\Gamma(\lambda+n)\, \Gamma(\lambda+n+\frac{3}{2})} = 2 \begin{cases} D_s\,\{f_1(\theta)\} & 0 \leqslant \theta < \alpha \\ D_s\,\{m(\theta)\} & \alpha < \theta < \beta \\ D_s\,\{g_1(\theta)\} & \beta < \theta \leqslant \pi, \end{cases}$$

where $D_c\{\ \ \}$ and $D_s\{\ \ \}$ stand for the operators $\frac{d}{d\theta}\{(\cos \theta/2)^{2\lambda}\{\ \ \}\}$ and

$\frac{d}{d\theta}\{(\sin \theta/2)^{2\lambda}\{\ \ \}\}$ respectively.

Using the orthogonality relation (2·1), we get

$$(5\cdot6)\quad C_n = -\frac{q_n\,\Gamma(\lambda+n+\frac{3}{2})}{\Gamma(\lambda+n+2)}\left[\int_0^{\alpha} D_c\{f_1(u)\} + \int_{\alpha}^{\beta} D_c\{m(u)\} + \int_{\beta}^{\pi} D_c\{g_1(u)\}\right] \times (\sin u/2)^{2\lambda+2} P_n^{(\lambda+1,\lambda-1)}(\cos u)\,du,$$

$$(5\cdot7)\quad C_n = \frac{q_n\,\Gamma(\lambda+n+\frac{3}{2})}{\Gamma(\lambda+n+2)}\left[\int_0^{\alpha} D_s\{f_1(u)\} + \int_{\alpha}^{\beta} D_s\{m(u)\} + \int_{\beta}^{\pi} D_s\{g_1(u)\}\right] (\cos u/2)^{2\lambda+2} P_n^{(\lambda-1,\lambda+1)}(\cos u)\,du.$$

If, in equations (5·1) and (5·2), we substitute for the coefficient C_n from (5·6) and (5·7) respectively we get, on interchanging the order of summation and integration, the equations

$$(5\cdot8)\quad -L_1(\theta) = \left[\int_0^{\alpha} D_c\{f_1(u)\} + \int_{\alpha}^{\beta} D_c\{m(u)\} + \int_{\beta}^{\pi} D_c\{g_1(u)\}\right] J_1(u,\theta)\,du,$$

$$(5\cdot9)\quad N_1(\theta) = \left[\int_0^{\alpha} D_s\{f_1(u)\} + \int_{\alpha}^{\beta} D_s\{m(u)\} + \int_{\beta}^{\pi} D_s\{g_1(u)\}\right] K_1(u,\theta)\,du,$$

where

$$(5\cdot10)\quad J_1(u,\theta) = \sum_{n=0}^{\infty} \frac{q_n}{\Gamma(\lambda+n+\frac{1}{2})\,\Gamma(\lambda+n+2)} (\sin u/2)^{2\lambda+2} P_n^{(\lambda+1,\lambda-1)}(\cos u)\, P_n^{(\lambda+\frac{1}{2},\lambda-\frac{1}{2})}(\cos\theta)$$

$$(5\cdot11)\quad K_1(u,\theta) = \sum_{n=0}^{\infty} \frac{q_n}{\Gamma(\lambda+n+\frac{1}{2})\,\Gamma(\lambda+n+2)} (\cos u/2)^{2\lambda+2} P_n^{(\lambda-1,\lambda+1)}(\cos u)\, P_n^{(\lambda-\frac{1}{2},\lambda+\frac{1}{2})}(\cos\theta).$$

With the help of (2·1), (2·2) and (2·3) it can be easily shown that

$$(5\cdot12)\quad J_1(u,\theta) = (2/\pi)^{\frac{1}{2}} (\cos\theta/2)^{1-2\lambda} (\cos\theta-\cos u)^{-\frac{1}{2}} H(u-\theta),$$

$$(5\cdot13)\quad K_1(u,\theta) = (2/\pi)^{\frac{1}{2}} (\sin\theta/2)^{1-2\lambda} (\cos u-\cos\theta)^{-\frac{1}{2}} H(\theta-u).$$

Thus equations (5·8) and (5·9) are equivalent to

$$(5\cdot14)\quad -(\pi/2)^{\frac{1}{2}} (\cos\theta/2)^{2\lambda-1} L_1(\theta) = \left[\int_{\theta}^{\alpha} D_c\{f_1(u)\} + \int_{\alpha}^{\beta} D_c\{m(u)\} + \int_{\beta}^{\pi} D_c\{g_1(u)\}\right] \frac{du}{(\cos\theta-\cos u)^{\frac{1}{2}}}\quad 0 \leqslant \theta < \alpha,$$

$$(5\cdot15)\quad (\pi/2)^{\frac{1}{2}} (\sin\theta/2)^{2\lambda-1} N_1(\theta) = \left[\int_0^{\alpha} D_s\{f_1(u)\} + \int_{\alpha}^{\beta} D_s\{m(u)\} \int_{\beta}^{\theta} D_s\{g_1(u)\}\right] \frac{du}{(\cos u-\cos\theta)^{\frac{1}{2}}}\quad \beta < \theta \leqslant \pi.$$

Since

$$\frac{d}{d\theta}\int_{\theta}^{\alpha} \frac{(\cos u/2)^{2\lambda} f_1(u)\sin u}{(\cos\theta-\cos u)^{\frac{1}{2}}}\,du = \frac{d}{d\theta}\left[\left\{2(\cos\theta-\cos u)^{\frac{1}{2}} f_1(u)(\cos u/2)^{2\lambda}\right\}_{\theta}^{\alpha}\right.$$

$$- 2 \int_\theta^\alpha D_c \{f_1(u)\} (\cos\theta - \cos u)^{\frac12} du]$$

$$= [- \frac{\sin\theta (\cos \alpha/2)^{2\lambda}}{(\cos\theta - \cos\alpha)^{\frac12}} f_1(\alpha) + \int_\theta^\alpha \frac{\sin\theta\, D_c \{f_1(u)\}}{(\cos\theta - \cos u)^{\frac12}} du]$$

we have

$$(5\cdot16) \qquad \int_\theta^\alpha \frac{D_c\{f_1(u)\}}{(\cos\theta - \cos u)^{\frac12}} du = \frac{(\cos\alpha/2)^{2\lambda}}{(\cos\theta - \cos\alpha)^{\frac12}} f_1(\alpha) +$$

$$\operatorname{cosec}\theta \frac{d}{d\theta} \int_\theta^\alpha \frac{(\cos u/2)^{2\lambda} f_1(u) \sin u}{(\cos\theta - \cos u)^{\frac12}} du$$

Similarly

$$(5\cdot17) \quad \int_\beta^\theta \frac{D_s\{g_1(u)\}}{(\cos u - \cos\theta)^{\frac12}} du = \operatorname{cosec}\theta \frac{d}{d\theta} \int_\beta^\theta \frac{(\sin u/2)^{2\lambda} g_1(u) \sin u}{(\cos u - \cos\theta)^{\frac12}} - \frac{(\sin\beta/2)^{2\lambda} g_1(\beta)}{(\cos\beta - \cos\theta)^{\frac12}}$$

After using (5·16) and (5·17), integrating by parts and using the fact that $f_1(\theta)$ and $g_1(\theta)$ are continuous at $\theta = \alpha$ and $\theta = \beta$ respectively, we obtain the equations

$$(5\cdot18) \quad P_1(\theta) = \frac{\sin\theta}{\pi}\left[(\pi/2)^{\frac12} (\cos\theta/2)^{2\lambda-1} L_1(\theta) + \tfrac12 \int_\alpha^\beta \frac{(\cos u/2)^{2\lambda} \sin u\, g_1(u)}{(\cos\theta - \cos u)^{3/2}} du\right]$$

$$= -\frac{1}{\pi} \frac{d}{d\theta} \int_\theta^\alpha \frac{(\cos u/2)^{2\lambda} \sin u\, f_1(u)}{(\cos\theta - \cos u)^{1/2}} du - \frac{\sin\theta}{2\pi} \int_\beta^\pi \frac{(\cos u/2)^{2\lambda} \sin u\, g_1(u)}{(\cos\theta - \cos u)^{3/2}} du, \quad 0 \leqslant \theta < \alpha,$$

$$(5\cdot19) \quad Q_1(\theta) = \frac{\sin\theta}{\pi}\left[(\pi/2)^{1/2} (\sin\theta/2)^{2\lambda-1} N_1(\theta) + \tfrac12 \int_\alpha^\beta \frac{(\sin u/2)^{2\lambda} \sin u\, g_1(u)}{(\cos u - \cos\theta)^{3/2}} du\right]$$

$$= \frac{1}{\pi} \frac{d}{d\theta} \int_\beta^\theta \frac{(\sin u/2)^{2\lambda-1} \sin u\, g_1(u)}{(\cos u - \cos\theta)^{1/2}} du - \frac{\sin\theta}{2\pi} \int_0^\alpha \frac{(\sin u/2)^{2\lambda-1} \sin u\, f_1(u)}{(\cos u - \cos\theta)^{3/2}} du,$$

$$\beta < \theta \leqslant \pi,$$

If we assume that

$$(5\cdot20) \qquad F_1(\theta) = -\frac{1}{\pi} \frac{d}{d\theta} \int_\theta^\alpha \frac{(\cos u/2)^{2\lambda} \sin(u)}{(\cos\theta - \cos u)^{1/2}} f_1(u)\, du,$$

$$(5\cdot21) \qquad G_1(\theta) = \frac{1}{\pi} \frac{d}{d\theta} \int_\beta^\theta \frac{(\sin u/2)^{2\lambda} \sin u}{(\cos u - \cos\theta)^{1/2}} g_1(u)\, du,$$

then we have

$$(5\cdot22) \qquad (\cos u/2)^{2\lambda} f_1(u) = \int_u^\alpha \frac{F_1(\theta)}{(\cos u - \cos\theta)^{1/2}} d\theta.$$

$$(5\cdot23) \qquad (\sin u/2)^{2\lambda} g_1(u) = \int_\beta^u \frac{G_1(\theta)}{(\cos\theta - \cos u)^{1/2}} d\theta.$$

The equations (5·18) and (5·19) can be written as

$$(5\cdot24) \quad P_1(\theta) = F_1(\theta) - \frac{\sin\theta}{2\pi} \int_\beta^\pi \frac{(\cot u/2)^{2\lambda} \sin u}{(\cos\theta - \cos u)^{3/2}} \left(\int_\beta^u \frac{G_1(v)\, dv}{(\cos v - \cos u)^{1/2}} \right) du,$$

$$0 \leqslant \theta < \alpha,$$

$$(5\cdot25)\quad Q_1(\theta) = G_1(\theta) - \frac{\sin\theta}{2\pi}\int_0^{\alpha} \frac{(\tan u/2)^{2\lambda}\sin u}{(\cos u - \cos\theta)^{3/2}}\left(\int_u^{\alpha}\frac{F_1(v)\,dv}{(\cos u - \cos v)^{1/2}}\right)du \qquad \beta < \theta \leqslant \pi.$$

On interchanging the order of integration we obtain

$$(5\cdot26)\qquad P_1(\theta) = F_1(\theta) - \int_{\beta}^{\pi} G_1(v)\,R_1(v,\theta)\,dv \qquad 0 \leqslant \theta < \alpha$$

$$(5\cdot27)\qquad Q_1(\theta) = G_1(\theta) - \int_0^{\alpha} F_1(v)\,S_1(v,\theta)\,dv \qquad \beta < \quad \leqslant \pi$$

where

$$(5\cdot28)\qquad R_1(v,\theta) = \frac{\sin\theta}{2\pi}\int_v^{\pi}\frac{(\cot u/2)^{2\lambda}\sin u\,du}{(\cos\theta - \cos u)^{3/2}(\cos v - \cos u)^{1/2}} \qquad 0 \leqslant \theta < \alpha$$

$$(5\cdot29)\qquad S_1(v,\theta) = \frac{\sin\theta}{2\pi}\int_0^{v}\frac{(\tan u/2)^{2\lambda}\sin u\,du}{(\cos u - \cos\theta)^{3/2}(\cos u - \cos v)^{1/2}} \qquad \beta < \theta \leqslant \pi$$

We can now eliminate $F_1(\theta)$ between equations (5·26) and (5·27) to obtain a Fredholm integral equation of the second kind for $G_1(\theta)$ or alternatively eliminate $G_1(\theta)$ between these equations to obtain a Fredholm integral equation for $F_1(\theta)$. If we eliminate $F_1(\theta)$ we obtain

$$(5\cdot30)\qquad Y_1(\theta) = G_1(\theta) - \int_{\beta}^{\pi} G_1(x)\,T_1(x,\theta)\,dx \qquad \beta < \theta \leqslant \pi,$$

where

$$(5\cdot31)\qquad Y_1(\theta) = Q_1(\theta) + \int_0^{\alpha} P_1(v)\,S_1(v,\theta)\,dv,$$

$$(5\cdot32)\qquad T_1(x,\theta) = \int_0^{\alpha} R_1(x,v)\,S_1(v,\theta)\,dv.$$

Alternatively

$$(5\cdot33)\qquad Z_1(\theta) = F_1(\theta) - \int_0^{\alpha} F_1(x)\,U_1(x,\theta)\,dx \qquad 0 \leqslant \theta < \alpha,$$

where

$$(5\cdot34)\qquad Z_1(\theta) = P_1(\theta) + \int_{\beta}^{\pi} Q_1(\theta)\,R_1(v,\theta)\,dv.$$

$$(5\cdot35)\qquad U_1(x,\theta) = \int_{\beta}^{\pi} R_1(v,\theta)\,S_1(x,v)\,dv.$$

Once $F_1(\theta)$ or $G_1(\theta)$ are known the other follows from (5·28) or (5·29). The functions $f_1(\theta)$ and $g_1(\theta)$ are obtained from (5·22) and (5·23) respectively and the coefficients C_n are determined from (5·6) or (5·7).

REFERENCES

1. Collins, W. D. On some triple series equations and their applications. *Archive Rational Mech. Anal.*, **11** : 122–137, (1962).
2. Erdelyi, A. Tables of integral transforms, *McGraw-Hill, New York*, 2 : (1954).
3. Srivastava, K. N. On dual series relations involving series of generalized Bateman K-functions. *Proc. Amer. Math. Soc* **17** : (1966).
4. Srivastava, K. N. A note on dual series relations involving series of Jacobi polynomials, (under communication).

INVERSIONS OF SOME INTEGRAL EQUATIONS

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ABSTRACT

In this paper inversions of three integral equations have been obtained. The first integral equation involves a Jacobi polyuomiual in the kernel. In inverting this finite Hankel transform and its inversion theorem (Sneddon I. N., (1951) Fourier Transforms, p. 83) have been utilised.

The second integral equation involves integro-exponential function $E_\nu(z)$ (Busbridge I. W., (1950), Quart. Journ Maths. (Oxford series), 1, 176–184). This integral equation is in the form of the complex convolution of the Laplace transform

The third integral equation involves a theta function in the kernel. As an example to the third theorem an integral representation of Dirac's delta function involving the product of two theta functions has been obtained.

1. INTRODUCTION

In this paper we give inversions of three integral equations. The first integral equation involves a Jacobi polynomial in the kernel. In inverting this integral equation we make use of finite Hankel transform and its inversion theorem [6, p. 83].

The second integral equation involves integro-exponential function $E_\nu(z)$, defined by Busbridge[3]. This function is used in problems connected with the radiative equilibrium of stellar atmosphares and in problem of neutron diffusion. The integral equation is in the form of the complex convolution. Complex convolution has been used by Srivastava[7] to solve a non-linear problem of circuit analysis.

The third integral equation involves a theta function in the kernel. The proof of this theorem is based on the convolution theorem of the Laplace transform. Among the convolution theorems of the different integral transforms the convolution theorem of the Laplace transform has the widest physical applications. With the help of this convolution, Mikusinski[5] has built an entire operational calculus.

As an example to the third theorem, we obtain an elegant integral representation of Dirac's delta function involving the product of two theta functions, which will provide wider applicability to the theta functions.

2. *Results required in the proof*:

We have due to Bhonsle [1, p. 189]

$$\int_0^1 y^{\alpha+1} (1-y^2)^\beta P_n^{(\alpha,\beta)}(1-2y^2) J_\alpha(xy)\, dy$$
$$= \frac{2^\beta \Gamma(\beta+n+1)}{n!\, x^{1+\beta}} J_{\alpha+\beta+2n+1}(x)$$
$$Re\ \alpha > -1,\ Re\ \beta > -1. \tag{2·1}$$

If $f(x)$ satisfies Dirichlet's conditions in the interval $o, a)$ and if its finite Hankel Transform in that range in defined to be

$$\bar{f}_J(\xi_i) = \int_0^a x f(x) J_\mu(x \xi_i)\, dx \tag{2·2}$$

where ξ_i is a root of the transcendental equation

$$J_\mu(a \xi_i) = 0 \tag{2·3}$$

then at any point of (o, a) at which the function $f(x)$ is continuous [6, p. 83]

$$f(x) = \frac{2}{a^2} \sum_i \bar{f}_J(\xi_i) \frac{J_\mu(x \xi_i)}{[J_\mu'(a \xi_i)]^2} \tag{2·4}$$

where the sum is taken over all the positive roots of the equation (2·3).

We shall represent the Laplace transform

$$F(p) = \int_0^\infty e^{-pt} f(t)\, dt, \; Re\, p > 0, \tag{2·5}$$

by

$$F(p) \doteq f(t) \tag{2·6}$$

Busbridge has defined the integro-exponential function $E_\nu(z)$, by

$$e^z E_\nu(z) = \int_0^\infty e^{-zu} (1+u)^{-\nu}\, du \tag{2·7}$$

so that

$$a^{1-\nu} e^{ap} E_\nu(ap) \doteq (a+t)^{-\nu}, \; a > o \tag{2·8}$$

We have [2, p. 157]

$$E_\nu(z) = e^{-\frac{1}{2}z} z^{\frac{1}{2}\nu-1} W_{-\frac{1}{2}\nu, \frac{1}{2}-\frac{1}{2}\nu}(z) \tag{2·9}$$

We have also [4, p. 131, p. 225, p. 137]

$$F_1(p) F_2(p) \doteq \int_0^t f_1(u) f_2(t-u)\, du \tag{2·10}$$

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_1(z) F_2(p-z)\, dz \doteq f_1(t) f_2(t) \tag{2·11}$$

$$\frac{l \sinh(x p^{\frac{1}{2}})}{\sinh(l p^{\frac{1}{2}})} \doteq \frac{\partial}{\partial x} \theta_3 \left(\frac{x+l}{2l} \,\middle|\, \frac{i \pi t}{l^2} \right), \; Re\, p > 0, \tag{2·12}$$

and

$$p^{-\nu-1} e^{ap} \Gamma(\nu+1, ap) \doteq (t+a)^\nu, \; |\arg a| < \pi, \; Re\, p > 0 \tag{2·13}$$

3. *Theorem* 1 :

If

$$\phi(x) = \int_x^1 \frac{1}{y} \left(\frac{x}{y}\right)^a \left(1 - \frac{x^2}{y^2}\right) P_n^{(\alpha,\beta)}\left(1 - \frac{2x^2}{y^2}\right) f(y)\, dy$$

$$Re\, \alpha > -1, \; Re\, \beta > -1 ; \tag{3·1}$$

then

$$f(y) = \frac{n!\, y^{\beta+1}}{2^{\beta-1} \Gamma(\beta+n+1)} \sum_i \frac{\bar{\phi}_J(\xi_i)\, \xi_i^{\beta+1} J_{\alpha+\beta+2n+1}(y \xi_i)}{[J'_{\alpha+\beta+2n+1}(y \xi_i)]^2} \tag{3·2}$$

where the sum is taken over all the positive roots of the transcendental equation

$$J_{\alpha+\beta+2n+1}(\xi_i) = 0 \tag{3·3}$$

and

$$\bar{\phi}_J(\xi_i) = \int_0^1 x\,\phi(x) J_\alpha(x\,\xi_i)\,dx \tag{3·4}$$

provided that

(*i*) $\phi(x)$ is absolutely continuous in the interval $(0, 1)$, and (*ii*) $\phi(1) = 0$

Proof: We have

$$\bar{\phi}_J(\xi_i) = \int_0^1 x\,\phi(x)\,J_\alpha(x\,\xi_i)\,dx$$

$$= \int_0^1 x J_\alpha(x\,\xi_i)\,dx \int_x^1 \frac{1}{y}\left(\frac{x}{y}\right)^\alpha \left(1-\frac{x^2}{y^2}\right)^\beta P_n^{(\alpha,\beta)}\left(1-\frac{2x^2}{y^2}\right) f(y)\,dy$$

$$= \int_0^1 f(y)\,dy \int_0^y \left(\frac{x}{y}\right)^{\alpha+1}\left(1-\frac{x^2}{y^2}\right)^\beta P_n^{(\alpha,\beta)}\left(1-\frac{2x^2}{y^2}\right) J_\alpha(x\,\xi_i)\,dx$$

Let $x/y = z$, then

$$\bar{\phi}_J(\xi_i) = \int_0^1 y f(y)\,dy \int_0^1 z^{\alpha+1}(1-z^2)^\beta P_n^{(\alpha,\beta)}(1-2z^2) J_\alpha(yz\,\xi_i)\,dz$$

and because of (2·1), we have

$$\int_0^1 y\, J_{\alpha+\beta+2n+1}(y\,\xi_i)\, y^{-\beta-1} f(y)\,dy$$

$$= \frac{n!\,\xi_i^{\beta+1}}{2^\beta\,\Gamma(\beta+n+1)}\,\bar{\phi}_J(\xi_i)$$

Now (3·2) follows from (2·4).

4. *Theorem 2* :

Let

$$F(p) = \frac{a^{1-\nu}}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{a(p-z)} E_\nu\{a(p-z)\} G(z)\,dz \tag{4·1}$$

then

$$G(p) = \frac{a^{1+\nu}}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{a(p-z)} E_{-\nu}\{a(p-z)\} F(z)\,dz \tag{4·2}$$

Proof:

If $G(p) \doteq g(t)$, then we make use of (2·11) to obtain the inverse transform of (4·1). Thus we obtain

$$f(t) = (a+t)^{-\nu} g(t) \tag{4·3}$$

so that

$$g(t) = (a+t)^\nu f(t). \tag{4·4}$$

Now the inverse Laplace transform of (4·4) will give

$$G(p) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{a(p-z)}(p-z)^{-\nu-1}\Gamma(\nu+1, a(p-z))F(z)\,dz$$

where [4, p. 387]

$$\Gamma(\nu+1, z) = \int_z^\infty e^{-t} t^\nu\,dt$$

$$= z^{\frac{1}{2}\nu} e^{-\frac{1}{2}z} W_{\frac{1}{2}\nu, \frac{1}{2}(\nu+1)}(z)$$

$$= z^{\nu+1} E_{-\nu}(z)$$

hence (4·2) follows.

Example :

Let

$$G(z) = E_\mu(az)\, a^{1-\mu}\, e^{az}$$

then

$$f(t) = (t+a)^{-(\nu+\mu)}$$
$$= a^{1-(\mu+\nu)}\, e^{ap}\, E_{\nu+\mu}(ap)$$

so that we have

$$E_{\nu+\mu}(a\,p) = \frac{a}{2\pi i}\int_{c-i\infty}^{c+i\infty} E_\nu\{a(p-z)\}\, E_\mu(az)\, dz$$

Making use of the theorem, we have

$$E_\mu(az) = \frac{a}{2\pi i}\int_{c-i\infty}^{c+i\infty} E_{-\nu}[a(p-z)]\, E_{\nu+\mu}(az)\, dz$$

5. *Theorem 3.*

If

$$g(t) \in c^1,\ o \leqq t < \infty\ ;\ g(o) = 0, \tag{5·1}$$

$$f(t) = \frac{1}{l}\int_0^t \frac{\partial}{\partial x}\,\theta_3\left(\frac{x+l}{2l}\,\Big|\,\frac{i\pi(t-u)}{l^2}\right) g(u)\, du \tag{5·2}$$

then

$$f(t) \in c^2\quad o \leqq t < \infty\ ; f(o) = f'(o) = 0\,, \tag{5·3}$$

$$g(t) = \frac{1}{x}\int_0^t \frac{d}{dl}\,\theta_3\left(\frac{x+l}{2x}\,\Big|\,\frac{i\pi(t-u)}{x^2}\right) f(u)\, du \tag{5·4}$$

We make use of (2·10) and (2·12) to prove this theorem.

Example :

First we prove that

$$\delta(t) = \frac{1}{xl}\int_0^t \frac{\partial}{\partial x}\,\theta_3\left(\frac{x+l}{2l}\,\Big|\,\frac{i\pi(t-u)}{l^2}\right)\frac{\partial}{\partial l}\,\theta_3\left(\frac{x+l}{2x}\,\Big|\,\frac{i\pi u}{x^2}\right) du$$

where $\delta(t)$ is the Dirac's delta function.

This is easily proved by taking Laplace transforms of both the sides.

The theorem will, therefore, give

$$\frac{\partial}{\partial l}\ \theta_3\left(\frac{x+l}{2x}\,\Big|\,\frac{i\pi t}{x^2}\right)$$
$$= \int_0^t \frac{\partial}{\partial l}\,\theta_3\left(\frac{x+l}{2x}\,\Big|\,\frac{i\pi(t-u)}{x^2}\right)\delta(u)\, du$$

This is self-evident.

REFERENCES

1. Bhonsle, B. R. On some results involving Jacobi polynomials *Proc. Indian Math. Soc.* **XXVI** : 187–190, (1962).
2. Bhonsle, B. R. On the Integro-exponential function. *Bull. Cal. Math. Soc.*, **49** : 157-162 (1957).
3. Busbridge, I. W. On the Intergo expenential function and the evaluation of some integrals involving it. *Quart. Jour. Maths. (Oxford second series)*, **1** : 176-184, (1950).
4. Erdelyi, A *et al*, Tables of Integral Transforms *McGraw-Hill Book Company Inc.* **I** : (1954).
5. Mikusinski Jan. Operational Calculus, *Pergamon Press*, (1959).
6. Sneddon, I. N. Fourier Transforms *McGraw-Hill Book Company, Inc.* (1951).
7. Srivastava, K. J. Complex convolution applied to non-linear problems. *Ganita*, **11** : 1–19, (1960).

SOME THEOREMS OF FRACTIONAL INTEGRATION

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ABSTRACT

The object of this paper is to establish connections between the Riemann-Liouville (fractional), Weyl (fractional) integrals and Varma transform. This has been done in the form of two theorems which includes as particular cases some known results. With the help of these theorems certain new fractional integrals containing confluent hypergeometric functions ψ_2, Ξ_2, and Ξ_1 have been derived.

INTRODUCTION

1. We call

$$\overset{+}{I_\mu} f(p) = R_\mu \{ f(t) ; p \} = \frac{1}{\Gamma(\mu)} \int_0^p f(t) (p-t)^{\mu-1} dt, \tag{1·1}$$

the Riemann-Liouville (fractional) integral of order μ, and

$$K_\mu^- f(p) = W_\mu \{ f(t) ; p \} = \frac{1}{\Gamma(\mu)} \int_p^\infty f(t) (t-p)^{\mu-1} dt \tag{1·2}$$

the Weyl (fractional) integral of order μ, of $f(t)$.

The classical Laplace transform

$$\varphi(p) = p \int_0^\infty e^{-pt} f(t) dt, \tag{1·3}$$

was generalized by Varma [4, p. 209] in the form,

$$\varphi(p) = p \int_0^\infty e^{-\frac{1}{2}pt} (pt)^{m-\frac{1}{2}} W_{k,m}(pt) f(t) dt. \tag{1·4}$$

When $k+m=\frac{1}{2}$, (1·4) reduces to (1·3) on account of the well known identity,

$$x^{m-\frac{1}{2}} W_{\frac{1}{2}-m,m}(x) = e^{-\frac{1}{2}x}$$

Throughout this paper we shall represent (1·3) and (1·4)) symbolically as, $L\{f(t) : p\} = \varphi(p)$ and $V_{k,m}\{f(t) : p\} = \varphi(p)$ respectively.

The object of this paper is to establish connections between the Riemann-Liouville (fractional) integral, Weyl (fractional) integral [3, p. 181] and Varma transform. This has been done in the form of the general theorems which include as particular cases some known results. These theorems have been further illustrated by means of suitable examples, so as to give few more fractional integrals containing confluent Hypergeometric functions ψ_2, Ξ_1, and Ξ_2. The following property of fractional integrals [3, p. 182] will be used in our investigations,

$$\int_0^\infty f_1(t) \overset{+}{I_\mu} f(t) dt = \int_0^\infty K_\mu^- f(t) f_2(t) dt, \tag{1·5}$$

where

$$W_\mu \{ f_1(t) ; p \} = K_\mu^- f(p) \text{ and } R_\mu \{ f_2(t) ; p \} = \overset{+}{I_\mu} f(p)$$

2. In this section we establish the relationship existing between Riemann-Liouville (fractional) integral and Varma transform. With the help of this relationship few Riemann-Liouville (fractional) integral of confluent hypergeometric functions have been derived.

Theorem 1.

If

$$R_\mu \{ f(t) ; p \} = \overset{+}{\underset{\mu}{I}} f(p) \tag{2·1}$$

then

$$p^{\lambda - \mu} V_{\frac{1}{2}(1-\lambda-\mu), \frac{1}{2}(\mu-\lambda)} \{ f(t) ; p \} = L \{ t^{-\lambda} \overset{+}{\underset{\mu}{I}} f(t) ; p \} \tag{2·2}$$

provided that the Riemann-Liouville (fractional) integral of $|f(t)|$ exist, $R(\mu) > 0$, $R(p) > 0$, $R(\mu - \lambda + \xi + 1) > 0$ and $R(\xi + 1) > 0$, where $f(t) = 0\,(t^\xi)$ for small 't'.

Proof :

We have [3, p. 202]

$$W_\mu \{ t^{-\lambda} e^{-at} ; p \} = a^{\nu - \frac{1}{2}} p^{-\nu - \frac{1}{2}} e^{-\frac{1}{2} ap} W_{\frac{1}{2} - \frac{\lambda}{2} - \frac{\mu}{2}, \frac{\lambda}{2} - \frac{\mu}{2}}(ap) \tag{2·3}$$

where, $R(ap) > 0$ and $R(\mu) > 0$. Using the relations (2·1) and (2·3) in the result (1·5) we obtain (2·2) after a little simplification.

If $\lambda \to 0$, then the theorem reduces to the well known result [3, p. 182], that if

$$R_\mu \{ f(t) ; p \} = \overset{+}{\underset{\mu}{I}} f(p)$$

then

$$p^{-\mu} L \{ f(t) ; p \} = L \{ \overset{+}{\underset{\mu}{I}} f(t) ; p \} \tag{2·4}$$

provided that the Riemann-Liouville (fractional) integral of $|f(t)|$ exist, $R(\mu) > 0$, $R(p) > 0$, $R(\mu + \xi + 1) > 0$ and $R(\xi + 1) > 0$ where $f(t) = 0\,(t^\xi)$ for small 't'.

Example 1.

If we start with,

$$f(t) = t^{\alpha - 1} \psi_2 (\alpha + \mu ; \gamma, \gamma' ; at, bt)$$

then [2, p. 223]

$$L \{ f(t) ; p \} = \Gamma(\alpha) p^{1-\alpha} F_4\left(\alpha, \alpha + \mu ; \gamma, \gamma' ; \frac{a}{p}, \frac{b}{p} \right) \tag{2·5}$$

for $R(\alpha) > 0$, $R(p) > 0$, $R(a)$, $R(b)$.

Using (2·5) in the result (2·4) and then obtaining inverse Laplace transform [2, p. 223], we get

$$R_\mu \{ t^{\alpha - 1} \psi_2 (\alpha + \mu ; \gamma, \gamma', at, bt) ; p \} = \frac{\Gamma(\alpha)}{\Gamma(\alpha + \mu)} \psi_2 (\alpha ; \gamma, \gamma ; ap, bp) \tag{2·6}$$

for $R(\alpha) > 0$ and $R(\mu) > 0$.

Burchnall and Chaundy [1, p. 124] have shown that,

$$\psi_2 (a ; c, c' ; x, x) = {}_3F_3\left(\begin{matrix} a, \frac{1}{2}(c+c'), \frac{1}{2}(c+c'-1) \\ c, c', c+c'-1 \end{matrix} ; 4x \right) \tag{2·7}$$

and hence if we take $a=b$ in (2·6), we obtain

$$R_\mu\left\{t^{\alpha-1}\,{}_3F_3\left(\begin{matrix}\alpha+\mu,\ \tfrac{1}{2}(\gamma+\gamma'),\ \tfrac{1}{2}(\gamma+\gamma'-1)\\ \gamma,\ \gamma',\ \gamma+\gamma'-1\end{matrix};4at\right);p\right\}=\frac{\Gamma(\alpha)}{\Gamma(\alpha+\mu)}\,p^{\alpha+\mu-1} \tag{2·8}$$

$$\times\ {}_3F_3\left(\begin{matrix}\alpha,\ \tfrac{1}{2}(\gamma+\gamma'),\ \tfrac{1}{2}(\gamma+\gamma'-1)\\ \gamma,\ \gamma',\ \gamma+\gamma'-1\end{matrix};4ap\right)$$

for $R(\alpha)>0$ and $R(\mu)>0$, which is a particular case of [3, p. 200].

Further if we put $\alpha+\mu=\gamma=\gamma'$ in (2·6) then [1, p. 126]

$$R_\mu\{t^{1-\alpha-2\mu}\exp(at+bt)\,I_{\alpha+\mu-1}(2t\sqrt{ab});p\}=\frac{\Gamma(\alpha)\,(ab)^{\alpha+\mu-1}}{\{\Gamma(\alpha+\mu)\}^2}\,p^{\alpha+\mu-1} \tag{2·9}$$

$$\times\ \psi_2(\alpha;\alpha+\mu,\alpha+\mu;ap,bp),$$

for $R(\alpha)>0$ and $R(\mu)>0$. Similarly if we put $\alpha=\gamma=\gamma'$ in (2·6) then we obtain [1, p. 126].

$$R_\mu\{t^{\alpha-1}\psi_2(\alpha+\mu;\alpha,\alpha;at,bt);p\}=\frac{\{\Gamma(\alpha)\}^2(ab)^{1-\alpha}}{\Gamma(\alpha+\mu)}\,p^{1+\mu-\alpha} \tag{2·10}$$

$$\times\exp(ap+bp)\,I_{\alpha-1}(2p\sqrt{ab},$$

for $R(\mu)>0$ and $R(\alpha)>0$.

Example 2.

If we start with, $f(t)=t^{\alpha'-1}\,\Xi_2(\alpha'+\mu,\beta,\gamma;a,bt)$ then [2, p. 223]

$$L\{f(t);p\}=\Gamma(\alpha')\,p^{1-\alpha'}\,\Xi_1\left(\alpha'+\mu,\alpha',\beta;\gamma;a,\frac{b}{p}\right), \tag{2·11}$$

$$R(\alpha')>0,\ R(p)>0,\ R(b),$$

Using (2·11) in (2·4) and then obtaining inverse Laplace transform [2, p. 223] we obtain

$$R_\mu\{t^{\alpha'-1}\,\Xi_2(\alpha'+\mu,\beta,\gamma;a,bt);p\}=\frac{\Gamma(\alpha')p^{\alpha'+\mu-1}}{\Gamma(\alpha'+\mu)}\ \Xi_2(\alpha',\beta'\ \gamma;a,bp), \tag{2·12}$$

for $R(\alpha')>0$, $R(\mu)>0$ and $|bp|<|$.

Example 3.

If we take

$$f(t)=t^{\beta'-1}\,\Xi_1(\alpha,\beta'+\mu,\beta,\gamma;a,bt)$$

then [2, p. 223]

$$L\{f(t);p\}=\Gamma(\beta')p^{1-\beta'}\,F_3\left(\alpha,\beta'+\mu,\beta,\beta';\gamma;a,\frac{b}{p}\right), \tag{2·13}$$

for $R(\beta')>0$, $R(p)>0$, $R(b)$. Using (2·13) in (2·4) and then obtaining inverse Laplace [2, p. 223] transform we obtain,

$$R_\mu\{t^{\beta'-1}\,\Xi_1(\alpha,\beta'+\mu,\beta,\gamma;a,bt);p=\frac{\Gamma(\beta')}{\Gamma(\beta'+\mu)}\,p^{\beta'+\mu-1} \tag{2·14}$$

$$\times\ \Xi(\alpha,\beta,\beta';\gamma;a,bp),$$

for $R(\beta')>0$, $R(\mu)>0$, and $|bp|<|$

3. In this section we establish the relationship existing between Weyl (fractional) integral and Varma transform. With the help of this relationship few Weyl (fractional) integrals of confluent hypergeometric functions have been derived.

Theorem 2.

If

$$W_\mu \{ f(\tfrac{1}{t}) ; p \} = K_\mu^- f(p) \tag{3·1}$$

then

$$p^\nu \, V_{\frac{1}{2}-\mu-\frac{\nu}{2}, -\frac{\nu}{2}} \{ t^{-\mu-1} f(t) ; p \} = L\{ t^{-\nu-1} K_\mu^- f(\tfrac{1}{t}) ; p \} \tag{3·2}$$

provided that the Weyl (fractional) integral of $| f(\tfrac{1}{t}) |$ exist,

$R(\mu) > 0$, $R(p) > 0$ and $R(\xi - \mu \pm \frac{\nu}{2} - \frac{\nu}{2}) > 0$, where $f(t) = 0\,(t^\xi)$ for small 't'.

Proof :

We have [3, p. 187]

$$R_\mu \{ t^{\nu-1} \exp\left(-\frac{a}{t} \right); p \} = a^{\frac{\nu}{2}-\frac{1}{2}} p^{-k} \exp\left(-\frac{a}{2p} \right) W_{\frac{1}{2}-\mu-\frac{\nu}{2}, \frac{\nu}{2}} \left(\frac{a}{p} \right), \tag{3·3}$$

$$R(\mu) > 0, \quad R\left(\frac{a}{p} \right) > 0.$$

Using (3·1) and (3·3) in the result (1·5), we obtain (3·2) after a little simplification.

If we put $\nu = -\mu$ in the above theorem on Varma transform we obtain the theorem on Laplace transform in the following form,

If

$$W_\mu \{ f(\tfrac{1}{t}) ; p \} = K_\mu^- f(p)$$

then

$$p^{-\mu} L \{ t^{-\mu-1} f(t) ; p \} = L \{ t^{\mu-1} K_\mu^- f(\tfrac{1}{t}) ; p \} \tag{3·4}$$

provided that the Weyl (fractional) integral of $| f(\tfrac{1}{t}) |$ exist, $R(\mu) > 0$, $R(p) > 0$ and $R(\xi - \frac{\mu}{2} \pm \frac{\mu}{2}) > 0$ where $f(t) = 0\,(t^\xi)$ for small 't'.

Example 1.

If we start with $f(t) = t^{\alpha-1} \psi_2 (\alpha - 1 ; \gamma, \gamma'; at, bt)$ then [2, p. 223]

$$L \{ t^{-\mu-1} f(t) ; p \} = \Gamma(\alpha - \mu - 1) \, p^{2+\mu-\alpha} F_4 \left(\alpha - \mu - 1, \alpha - 1; \gamma, \gamma'; \frac{a}{p}, \frac{b}{p} \right),$$

$$R(\alpha - \mu - 1) > 0, \; R(p) > 0, \; R(a), R(b)$$

Using (3·5) in the result (3·4) and then obtaining the inverse Laplace transform we obtain,

$$W_\mu\{t^{1-\alpha}\,\psi_2\left(\alpha-1\,;\,\gamma,\,\gamma';\,\frac{a}{t},\,\frac{b}{t}\right);p\}=\frac{\Gamma(\alpha-\mu-1)}{\Gamma\,(\alpha-1)}\,p^{1+\mu-\alpha}$$

$$\times\,\psi_2\left(\alpha-\mu-1;\,\gamma,\gamma'\,;\,\frac{a}{p},\,\frac{b}{p}\right), \tag{3·6}$$

for $R(\alpha-\mu-1)>0$, $R(\mu)>0$ and $R(p)>0$. Using (2·7) we get, when $a=b$

$$W_\mu\,\{t^{1-\alpha}\,{}_3F_3\left(\begin{matrix}\alpha-1,\,\frac{1}{2}(\gamma+\gamma'),\,\frac{1}{2}(\gamma+\gamma'-1)\\ \gamma,\,\gamma',\,\gamma+\gamma'-1\end{matrix}\,;\,\frac{4a}{t}\right);p\}=\frac{\Gamma(\alpha-\mu-1)}{\Gamma\,(\alpha-1)}\,p^{1+\mu-\alpha}$$

$$\times\,{}_3F_3\left(\begin{matrix}\alpha-\mu-1,\,\frac{1}{2}\,(\gamma+\gamma'),\,\frac{1}{2}\,(\gamma+\gamma'-1)\\ \gamma,\,\gamma',\,\gamma+\gamma'-1\end{matrix}\,;\,\frac{4a}{p}\right), \tag{3·7}$$

for $R\,(\alpha-\mu-1)>0$, $R(\mu)>0$ and $R\,(p)>0$, which is a particular case of [3, p. 212].

Further if we put, $\alpha-1=\gamma=\gamma'$, in (3·6) then [1, p. 126]

$$W_\mu\,\{t^{\mu-3}\,\exp\left(\frac{a}{t}+\frac{b}{t}\right)I_{\alpha-2}\left(\frac{2\,\sqrt{ab}}{t}\right);p\,\}=\frac{\Gamma(\alpha-\mu-1)}{\{\Gamma(\alpha-1)\}^2}\,(ab)^{\alpha-2}\,p^{1+\mu-\alpha}$$

$$\times\,\psi_2\left(\alpha-\mu+1;\,\gamma,\,\gamma';\,\frac{a}{p},\,\frac{b}{p}\right) \tag{3·8}$$

for $R\,(\alpha-\mu-1)>0$, $R\,(\mu)>0$ and $R\,(p)>0$.

Similarly if we put, $\alpha-\mu-1=\gamma=\gamma'$, in (3·6) then we obtain [1, p. 126]

$$W_\mu\,\{t^{1-\alpha}\,\psi_2\left(\alpha-1\,;\,\alpha-\mu-1,\,\alpha\quad\mu-1\,;\frac{a}{t},\,\frac{b}{t}\right);p\,\}=\frac{\{\Gamma(\alpha-\mu-1)\}^2}{\Gamma(\alpha-1)}(ab)^{\mu-\alpha}$$

$$\times\,p^{\alpha-\mu+1}\,\exp\left(\frac{a}{p}+\frac{b}{p}\right)I_{\alpha-\mu}\left(\frac{2\,\sqrt{ab}}{p}\right), \tag{3·9}$$

for $R\,(\alpha-\mu-1)>0$, $R\,(\mu)>0$ and $R\,(p)>0$.

Example : 2.

If we start with, $f\,(t)=t^{\alpha'-1}\,\Xi_2\,(\alpha'-1,\,\beta,\,\gamma\,;\,a,\,bt)$ then [2, p. 223]

$$L\,\{t^{-\mu-1}\,f\,(t)\,;\,p\}=\Gamma(\alpha'-\mu-1)\,p^{2+\mu-\alpha'}\,\Xi\left(\alpha'-1,\,\alpha'-\mu-1,\,\beta,\gamma\,;\,a,\frac{b}{p}\right), \tag{3·10}$$

for $R\,(\alpha'-\mu-1)>0$, $R\,(p)>0$, $R\,(b)$. Using (3·10) in (3·4) and then obtaining inverse Laplace transform [2, p. 223] we get,

$$W_\mu\,\{t^{1-\alpha'}\,\Xi_2\left(\alpha'-1,\,\beta,\,\gamma\,;\,a,\,\frac{b}{t}\right);p\,\}=\frac{\Gamma(\alpha'-\mu-1)}{\Gamma(\alpha'-1)}\,p^{1+\mu-\alpha'} \tag{3·11}$$

$$\times\,\Xi_2\left(\alpha'-\mu-1,\,\beta,\,\gamma\,;\,a,\frac{b}{p}\right),$$

for $R\,(\alpha'-\mu-1)>0$, $R\,(\mu)>0$ and $R\,(p)>0$, $R(b)$.

Example 3.

If we start with $f(t) = t^{\beta'-1} \Xi_1(\alpha, \beta'-1, \beta ; \gamma ; a, bt)$ then [2, p. 223]

$$L\{t^{-\mu-1} f(t) ; p\} = \Gamma(\beta'-\mu-1)\, p^{2+\mu-\beta'} F_3\left(\alpha, \beta'-1, \beta, \beta'-\mu-1, \gamma; a, \frac{b}{p}\right), \quad (3\cdot12)$$

for $R(\beta'-\mu-1) > 0$, $R(p) > 0$, $R(b)$. Using (3·12) in (3·4) and then obtaining inverse Laplace transform [2, p. 223] we get,

$$W_\mu\left\{t^{1-\beta'} \Xi_1\left(\alpha, \beta'-1, \beta, \gamma ; a, \frac{b}{t}\right) ; p\right\} = \frac{\Gamma(\beta'-\mu-1)}{\Gamma(\beta'-1)} p^{1+\mu-\beta'}$$

$$\times\, \Xi_1\left(\alpha, \beta, \beta'-\mu-1; \gamma; a, \frac{b}{p}\right), \quad (3\cdot13)$$

for $R(\beta'-\mu-1) > 0$, $R(\mu) > 0$ and $R(p) > 0$, $R(b)$.

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REFERENCES

1. Burchnall, J. L. and Chaundy, T. W. Expansions of Appell's double hypergeometric functions. *Quart J. Math. Oxford*, 12 : 112–128, (1941).
2. Erdelyi, A. *et al.*, Tables of integral transforms. *McGraw-Hill New York*, I : (1954).
3. Erdelyi, A. *et al.*, Tables of integral transforms. *McGraw-Hill, New York*, II : (1954).
4. Varma, R. S. On a generalisation of Laplace integral. *Proc. Nat. Acad Sci. India*, 20A. : 209–216, (1951).

INFLUENCE OF BLUE GREEN ALGAE ON COMPOSTING OF MUNICIPAL WASTE

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ABSTRACT

Influence of blue green algae (Anabeana naviculoides) on composting of municipal waste was investigated. By the inoculation of algae some saving of carbon and small increase in nitrogen was observed. Algae seemed to have some stimulating effect on solubilisation of phosphate also. Fixation of nitrogen due to inoculation of algae formed only a small fraction of the nitrogen fixed by the slow oxidation of organic matter aided by phosphate.

Nitrogen fixations by blue green algae is specially interseting in that these organisms assimilate carbondioxide and thus are independent of organic compounds and combined nitrogen as well. Consequently they are supposed to be the most completely autotropic organisms known. Light is indispensable for them. Organic matter and phosphate have been reported to be helpful in their growth and activity (Pearsal, 1932 ; Okuda and Yamaguchi, 1952 ; De and Sulaiman, 1942) The present investigations were undertaken to study the influence of blue green algae (Anabaena naviculoides) both in presence and absence of phosphate on carbon nitrogen changes and availability of phosphate during the composting of municipal waste.

METHODS AND MATERIALS

The following medium was used for growing culture of Anabaena naviculoides :

KNO_3	0·2 gm.
K_2HPO_4	0·2 gm.
$MgSO_4 . 7H_2O$	0·2 gm.
$CaCl_2$	0·1 gm.
$FeCl_3$ (1%)	2 drops.
Distilled water	1000 cc.

The composting of municipal waste was carried in small shallow heaps. Phosphate was added in the form of Tata basic slag at the rate of 65 lbs. P_2O_5 per ton of organic matter. The heaps were inoculated with Anabaena culture grown in the above medium on 22nd August, 1963. The trials were conducted in duplicate. After definite intervals of time composite samples were taken out powdered, sieved and analysed on oven dry basis. Total carbon was determined by the method of Robinson, Mcleans and Williams (1929) and total nitrogen by Kjeldahl salicyllic acid method (John-Brooks, 1936). Ammoniacal and nitrate nitrogen were determined by Olsens method as modified by Richardson (Piper, 1942) and available phosphate by Dyer's method (1934).

RESULTS AND DISCUSSION

Analysis of Tata basic slag %

SiO_2	..	33·64
Fe_2O_3	..	6·95
K_2O	..	0·907
CaO	..	30·36
MgO	..	2·287
Total P_2O_1	..	7·906
Available P_2O_5	..	4·061

TABLE I

Effect of blue green algae in presence and absence of basic slag on carbon content of Municipal Compost

Date of Sampling	FIRST TRIAL				SECOND TRIAL			
	Un-Phosphated municipal waste		Phosphated municipal waste		Un-Phosphated municipal waste		Phosphated municipal waste	
	Uninoculated	Inoculated	Uninoculated	Inoculated	Uninoculated	Inoculated	Uninoculated	Inoculated
20-6-63	12·938	13·016	12·981	12·967	13·140	13·192	13·120	13·121
20-7-63	10·734	10·764	10·086	10·071	10·926	10·911	10·225	10·226
20 8-63	9·204	9·435	8·086	8·295	9·394	9·265	8·224	8·413
20-9-63	8·136	8·317	6·856	7·104	8·325	8·531	6·988	7·296
20-10 63	7·336	7·719	5·995	6·345	7·526	7 867	6·126	6·561
20-11-63	6·737	7·223	5·301	5·734	6·296	7·302	5·432	5·952

TABLE II

Effect of blue green algae in presence and absence of basic slag on Nitrogen content of Municipal Compost

20-6-63	0·5210	0 5221	0·5218	0·5208	0·5214	0·5244	0 5244	0·5165
20-7 66	0·6197	0 6212	0·7009	0·7001	0·6155	0·6184	0·7101	0·7061
20-8-63	0·6881	0·6962	0·8308	0·8379	0·6871	0·6947	0 8425	0·8498
20-9-63	0 7393	0·7493	0·9301	0·9472	0·7453	0·7549	0·9420	0·9599
20 10-63	0·7791	0·7958	1·0014	1·0248	0·7805	0·7947	1·0052	1·0252
20-11-63	0·7563	0·7767	0·9746	1·0061	0·7573	0·7768	0·9776	1·0088

TABLE III

Effect of blue green algae in presence and absence of basic slag on $NH_3 - N$ content of Municipal Compost

Date of Sampling	FIRST TRIAL				SECOND TRIAL			
	Un-Phosphated municipal waste		Phosphated municipal waste		Un-Phosphated municipal waste		Phosphated municipal waste	
	Uninoculated	Inoculated	Uninoculated	Inoculated	Uninoculated	Inoculated	Uninoculated	Inoculated
20-6-63	0·0108	0 0113	0·0118	0 0116	0·0126	0·0121	0·0120	0·0126
20-7-63	0·0215	0·0211	0·0279	0·0276	0·0226	0·0221	0·0292	0·0293
20-8-63	0·0288	0·0411	0·0402	0·0290	0·0283	0·0283	0·0440	0·0432
20-9-63	0 0341	0·0330	0·0492	0·0484	0·0332	0·0324	0·0502	0·0490
20-10-63	0·0376	0·0369	0·0537	0·0534	0·0370	0·0365	0·0541	0·0534
20-11-63	0·0344	0·0348	0·0491	0·0501	0·0330	0·0339	0·0496	0·0508

TABLE IV

Effect of blue green algae in presence and absence of basic slag on $NO_3 - N$ content of Municipal Compost

20 6-63	0·0161	0·0160	0·0165	0·0170	0·0190	0·0180	0·0176	0·0174
20-7-63	0·0309	0·0306	0·0442	0·0445	0·0324	0·0327	0·0441	0·0448
20-8-63	0·0443	0·0425	0·0695	0·0680	0·0448	0·0428	0·0671	0·0659
20-9 63	0·0565	0·0545	0·0869	0·0851	0·0556	0·0537	0·0873	0·0856
20-10-63	0·0631	4·0615	0·0996	0·0980	0·0639	0·0626	0·0999	0·0987
20-11-63	0 0591	0·0588	0·0939	0·0931	0·0594	0·0589	0·0982	0·0918

TABLE V

Effect of blue green algae in presence and absence of basic slag on Available P_2O_5 content of Municipal Compost

20-6-63	0·1135	0·1142	0·1125	0·1129	0·1275	0·1261	0·1270	0·1246
20-7-63	0·1462	0·1480	0·1790	0·1768	0·1602	0·1598	0·1867	0·1852
20-8-63	0·1715	0 1729	0·2267	0·2273	0·1781	0 1757	0·2309	0·2325
20-9-63	0·1883	0·1990	0·2536	0·2580	0·1891	0·1901	0·2651	0·2691
20-10-63	0·1982	0·2024	0·2671	0·2749	0·1965	0·2001	0·2756	0·2817
20-11-63	0·2016	0·2079	0·2768	0·2859	0·2002	0·2064	0·2815	0·2910

The results recorded in foregoing tables show that there is considerable oxidation of carbonaceous compounds when municipal waste is allowed to undergo slow oxidation in air and that when phosphate in the form of Tata basic slag is also incorporated, the velocity of oxidation is appreciably increased. This may be due to the alkaline nature of the basic slag and alkalinity favours

oxidation is a well known fact. During the composting process a progressive increase in nitrogen concentration has also been observed. The increase in nitrogen is more pronounced in systems containing slag. This is due to the fact that more nitrogenous compounds are created from the air per unit weight of the carbon oxidised in the presence of basic slag than its absence (Dhar, 1952).

It is further observed that when municipal waste is inoculated with Anabaena naviculoides, there is some saving of carbon and shall increase in nitrogen. This seems to be due to addition of organic matter by the growth of algae and its consequent oxidation and fixation of nitrogen. Moreover in presence of phosphates growth and activity of algae seems to be further enhanced. This is in line with the finding of Okuda and Yamaguchi (1952) who reported increased growth and activity of algae in Japanese rice soils on the application of calcium carbonate and dipotassium hydrogen phosphate. Dey and Sulaiman (1942) also observed that phosphate and calcium supply appears to be the important limiting factors for the growth of blue green algae.

The experimental results also show that available phosphate of the system increased with the oxidation of carbon and it is greatly enhanced when phosphate is added. In the algal sets the increase in availability of phosphate is slightly greater in the sets without algae. This increase in availability of phosphate appears to be due to the production of more carbonic acid and organic acid and also to dephosphorylation of the organic phosphorous of the nucleic acid and nucleotides etc., of the algal material (Thompson and Black, 1950).

It is fascinating to record that the increase in nitrogen fixation due to inoculation of algae is very small as compared to nitrogen fixed by the slow oxidation of organic matter aided by phosphates. Similarly, Gupta (1954) has experimentally shown that algal contribution in enriching soils is very small as compared to that of the organic matter undergoing oxidation.

REFERENCES

1. De. P. K, and Sulaiman, M. *Ind. J. Agri. Sci.* **22** : 375 (1942).
2. Dhar, N. R. *Presi. Add Nat. Acad. Sci., India*, **47** : (1952).
8. Dyer, B *Trems. Chem. Soc.*, **65** ; 115 (1934)
4. Gupta, G. P. *D. Phil. Thesis, Allahabad University* (1954).
5. John Brooks, R. St. *Sec. Internatl. Cong. Microb., London*, (1936), Report of the proceedings, *Harrisrn and Sons, Ltd., London* **34**, (1935),
6. Okuda, A. and Yamaguchi, M. *Mem. Res. Inst. Food Sci Kyoto Univ.* **4** : 1 (1952).
7. Pearsal, W. H. *J. Ecol.*, **20** : 241 (1932)
8. Piper, C. S. *Soil and Plant Analysis, Univereity of Adelaide*, 208, (1942).
9. Robinson, G. W., Mcleans, W. and Williams, R. *J. Agri. Sci*, **29** : 315 - 324 (1929).
10. Thompson, L. M. and Black, C. A. *Soil Sci Soc. Amer. Proc.*, **14** : 147, (1950).

INFLUENCE OF DIRECT APPLICATION OF ORGANIC MATERIALS AND BASIC SLAGS ON YIELD OF PADDY AND WHEAT CROPS AND SOIL FERTILITY

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ABSTRACT

In the present investigation, the effect of different organic materials like wheat straw, mixture of grasses and paddy straw alone and in conjunction with Tata basic slag and German basic slag on the yield of paddy and wheat crop and also on soil fertility were studied It was found that by incorporation of all types of organic materials and basic slags much higher crop yields are obtained and at the same time it was also observed that land fertility can be considerably increased by this method.

The outstanding feature of Indian soils is that they are not only deficient in nitrogen and other materials but also in organic matter. Dhar and Mukerjee (1935) analysed many soils from various parts of India and found that most of the Indian soils contain about 0·04% total nitrogen. Thus the problem of increasing the nitrogen status of our soils is of foremost importance. Artificial fertilizers are of little use as they do not improve the nitrogen status of the soils. It, therefore, becomes quite desirable that organic manures may be used to supplement the essential nutrients lacking in our soils and build up soil fertility.

One strong school of opinion is gaining ground who doubt the wisdom of composting and advocate the direct application of plant materials to the land. Dhar (1952) in his presidential address to the National Academy of Sciences, India, has emphasized that the direct application of plant materials to the fields before composting is more beneficial to crops, because, the energy materials like the carbohydrates, lignins, fats etc. when added to the soil, are partially oxidised and in this process nitrogen of the air is fixed. Hence, much more humus (which is a combination of protein with lignin or cellulose or carbobydrate mixed with microorganisms) is formed and added in the soil when plant materials are mixed in the soil direct instead of their addition after composting them elsewhere. The method of direct application of plant materials to soil without composting has been adopted in farms in Sweden, England and Pennsylvania and California, U. S. A. It has been reported in Rothamsted (1935–39) that under favourable conditions uncomposted refuse material could be directly applied to the land with beneficial result on plant growth equal to, and in some cases even superior to, those obtained by the application of compost manure prepared from refuse.

That phosphate fertilizers improve the fertility of the soil apart from increasing yield, especially that deficient in available phosphorus, is an universally acclaimed fact. It is no wonder, therefore, that extensive reserach work is being carried on the use of phosphates especially basic slags, in land fertility improvement all over the world, particularly in Europe. A number of excellent reviews (Volkaerts, 1954 ; Schmitt, 1954, Glastra, 1955, Gericke, 1955) are available. In

acid soils basic slags has been found superior to super-phosphate (Franck 1954) or equivalent to hyperphosphate or rockphosphate. Lask (1954), Weihling (1959), Balfour (1961), Bell (1952) and Barmann (1957) reported beneficial effect of basic slag on crop yield.

In the present investigation field trials were conducted on a farmers field (Village–Jograypur, Allahabad) inorder to study the influence of different organic materials like wheat straw, mixture of grasses and paddy straw alone and in combination with Tata basic slag and German basic slag on the yield of paddy grain, paddy straw, wheat grain and wheat straw and soil fertility.

METHODS AND MATERIALS

The field selected for the experiment was more or less uniform. The experience of the resident farmer was utilized in the selection of the site. The soil was of the alluvial type in which crops were grown for a number of years. The field was ploughed and amendments were mixed with the soil on 14-6-1962. The field was kept moistened with water till the rain started and six ploughings were given during the decompositions of organic matter. Paddy was transplanted on 3-8-1962. After harvesting of the paddy crop the field was again prepared and wheat was sown. Wheat was harvested on 26-3-1963. Paddy was again sown in the field without any further treatment. Canal water was used for irrigation of the paddy and wheat crops.

Design of the experiment was Randomised Block lay out and area of each plot was 1/60th of an acre. The details of the treatments are as under :

T_1– Soil alone (Control)

T_2– Soil+Wheat straw @ 10 tons per acre.

T_3– Soil+Wheat straw @ 10 tons per acre+60 lbs. P_2O_5 per acre as Tata basic slag.

T_4– Soil+Wheat straw @ 10 tons per acre+60 lbs. P_2O_5 per acre as German basic slag.

T_5– Soil+grasses @ 10 tons per acre.

T_6– Soil+grasses @ 10 tons per acre+60 lbs. P_2O_5 per acre as Tata basic slag.

T_7– Soil+grasses @ 10 tons per acre+60 lbs. P_2O_5 per acre as German basic slag.

T_8– Soil+paddy straw @ 10 tons per acre.

T_9– Soil+paddy straw @ 10 tons per acre+60 lbs. P_2O_5 per acre as Tata basic slag.

T_{10}– Soil+paddy straw @ 10 tons per acre+60 lbs. P_2O_5 per acre as German basic slag.

T_{11}– Soil+Tata basic slag at 60 lbs. per acre.

T_{12}– Soil+German basic slag @ 60 lbs. per acre.

The soil samples (0″–6″) were taken before mixing the amendments and harvesting of the last paddy crop and analysed.

ABBREVIATIONS

T. B. S. for **Tata basic slag.**
G. B. S. for **German basic slag.**

RESULTS AND DISCUSSIONS

Analysis of the soil under field trial

		HCl extract determination	
Moisture	1·63 %	Total HCl insoluble	81·6130 %
Loss on ignition	3·9153 %	Sesquioxide	11·2071 %
Total nitrogen	0·0553 %	Fe_2O_3	4·2630 %
Organic Carbon	0·5648 %	CaO	0·9863 %
NH_3–nitrogen	0·0041 %	P_2O_5	0·0815 %
NO_3–nitrogen	0·0061 %	K_2O	0·9578 %
Available P_2O_5	0·0164 %	MgO	0·5990 %
pH	7·8		
C. E. C.	12·8 Mc/100 gm soil		

Mechanical Analysis

Sand	63·9 %
Sil	15·0 %
Clay	20·5 %

Analysis of organic materials utilised (%)

Composition	Wheat straw	Mixture of grasses	Paddy straw
Total carbon	36·6124	35·7274	34·3135
Total nitrogen	0·6024	0·6001	0·5368
C/N	60·7	59·5	64·1
CaO	0·7562	0·8325	0·7053
P_2O_5	0·4869	0·4727	0·3987
K_2O	0·6945	0·7126	0·6641

Analysis of basic slags

	Tata basic slag %	*German basic slag %*
SiO_2	33·64	22·75
Fe_2O_3	6·95	16·20
K_2O	0·907	0·952
CaO	30·367	33·55
MgO	2·287	5·50
Total P_2O_5	7·908	17·905
Available P_2O_5	4·061	9·396

DISCUSSIONS

A perusal of the preceding results reveals that by the addition of different organic materials like wheat straw, mixture of grasses and paddy straw to the field better yields are obtained than in the control set in which no organic matter has been added. Moreover, when phosphates were incorporated with these organic materials much better results were obtained than the treatments in which only organic matter was added. The order of efficiency of different treatments in increasing the yield of paddy and wheat crops is as under :

Prosphated wheat straw > Phosphated grasses > Phosphated Paddy Straw > Wheat Straw > Grasses > Paddy Straw > German basic slag > Tata basic slag > Control.

The results recorded in Table I show that application of organic materials and slags both in combination and separately gave significantly higher yield of grain and straw of the first paddy and the following wheat crop over control. In case of last paddy crop, however, all the treatments except slags alone recorded significantly higher yield over control at both the levels of significance. The yield difference in case of two types of slags when applied alone were not significant. Application of phosphate along with organic matter recorded significantly higher yield than organic matter alone.

The 2nd crop of paddy gave lower yield of grain and straw as compared to the 1st paddy crop. This is because the preceding crops of paddy and wheat have taken up certain part of the plant nutrients. Similarly, Russell (1951) reported that a dressing of dung given once in four years produced a marked effect in its first year and less but quite distinct effect in the second, third and fourth year.

TABLE I

Yield of paddy and wheat crops in Kg/Hec.

	1st Paddy Crop		Wheat Crop		IInd Paddy Crop	
	Paddy Grain	Paddy Straw	Wheat Grain	Wheat Straw	Paddy Grain	Paddy Straw
T_1 =	1567·9	4075·5	1302·6	2993·6	1513·1	3834·6
T_2 =	2613·1	7365·5	2080·7	4994·3	2043·6	5112·9
T_3 =	3383·4	9514·4	2907·6	6772·7	2537·3	5928·0
T_4 =	3673·9	9855·3	3118·1	7335·9	2599·4	6194·8
T_5 =	2563·9	7300·6	1957·7	4727·6	1951·8	4933·1
T_6 =	3340·4	9266·9	2778·8	6372·6	2402·3	5631·6
T_7 =	3571·6	9618·2	2958·1	6772·7	2457·2	4401·5
T_8 =	2420·1	6550·4	1827·3	4475·6	1932·5	4520·1
T_9 =	3001·5	9040·2	2531·3	6002·1	2326·2	5335·2
T_{10} =	3235·2	9336·6	2716·5	6624·5	2389·0	5705·7
T_{11} =	1790·3	4772·0	1482·0	3453·1	1674·7	4045·9
T_{12} =	1834·7	4920·4	1562·0	3675·4	1693·8	4149·6
C. D. at 5%	154·12	284·54	228·23	240·08	213·40	326·04
C. D. at 1%	129·46	224·19	191·71	201·67	179·26	272·38

TABLE II

Composition of soil after harvesting of last paddy crop (%)

	TREATMENT											
	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}
	Soil alone (Control)	Soil+ Wheat straw	Soil+ Wheat straw+ TBS	Soil+ Wheat straw+ GBS	Soil+ Grasses	Soil+ grasses+ TBS	Soil+ grasses+ GBS	Soil+ paddy straw	Soil+ paddy straw+ TBS	Soil+ paddy straw+ GBS	Soil+ TBS	Soil+ GBS
Total Carbon	0·5573	0·6530	0·6392	0·6407	0·6390	0·6204	0·6232	0·6201	0·6149	0·6180	0·5501	0·5556
Total Nitrogen	0·0547	0·0611	0·0665	0·0680	0·0605	0·0645	0·0650	0·0591	0·0643	0·0643	0·0550	0·0551
NH_3 Nitrogen	0·0038	0·0044	0·0050	0·0055	0·0042	0·0051	0·0051	0·0042	0·0048	0·0050	0·0037	0·0036
NO_3 Nitrogen	0·0057	0·0062	0·0070	0·0070	0·0065	0·0071	0·0068	0·0058	0·0070	0·0069	0·0059	0·0059
Available P_2O_5	0·0166	0·0221	0·0342	0·0351	0·0211	0·0350	0·0362	0·0215	0·0341	0·0338	0·0182	0·0199
C/N	10·1	10·6	9·6	9·4	10·5	9·5	9·5	10·3	9·5	9·6	10·6	10·1

The results recorded in Table 2 show that in spite of the large removal of nutrients by the raised crops from the plots where organic materials and phosphate were applied, they still contained greater amounts of total nitrogen, available nitrogen, available phosphate and total carbon. The reasons for this remarkable effect of organic materials and phosphates is not far to seek. Dhar and Coworkers (1961) in a large number of experiments extended to several years have found that all types of organic substances like cow dung municipal waste, grasses, straw and other plant residue etc. and phosphates, when incorporated in soil can build up soil fertility permanently by fixing atmospheric nitrogen and supplying available nitrogen, phosphate, potash, trace element and humus. Similarly, Karraker working in U. S. A. obtained the results showing marked fixation of nitrogen by a mixture of manure and phosphate :—

Average of the three field treatments	Nitrogen in soil (lbs/acre)	Corn yield (Bushel per acre)
No menure	1600	17
Manure	1760	36
Manure+Phosphate	1990	51

Recently, Lady Eve Balfour (1961) of the Soil Association, New Bells Farm, Suffolk, England reported that the plot to which basic slag was added to the straw, contained the largest amount of nitrogen and produced the biggest crop (30·4 Cwt. barley grain per acre) and the land to which 112 lbs. of nitrogen was added yielded 22·8 Cwt. barley grain per acre while the control plot produced only 14 Cwt. barley grain per acre. Similar results were obtained at Burdwan, Mindapore and Birbhum, Bengal, India (Ghosh, 1963) by the incorporation of straw and Phosphate together. Bell (1952) and Barmann (1957) also observed remarkable effect of basic slag in increasing crop yield.

From the foregoing field trials, it is evident that ploughing in of organic materials especially wheat straw along with basic slag—a by product of the steel industry, certainly increases the land fertility permanently and increased yields are obtained.

REFERENCES

1. Balfour Lady Eve Compare Dhar (*Presi. Addr. to the 48th Indian Science Congress*, 30, (1961).
2. Barmann, C. *Phosphorsaure*, 17 : 301-315, (1957).
3. Bell, J. E. *N. Z. J. Agri.*, 85 : 327-328, (1952).
4. Dhar, N. R and Mukerjee, S. K. *J. Ind. Chemi. Soc.*, 12 : 17, (1935).
5. Dhar, N. R. Business Matter, *Nat. Acad Sci. India*, (1952).
6. Dhar, N. R. *Presi. Addr.* 48th *Indian Sci. Cong.* (1961).
7. Franck, O. *Phosphorsaure*, 14 : 259-262, (1954).
8. Gericke, S. and Barmann, C. *Phosphorsaure*, 17 : 185-196, (1955).
9. Glastra, J. L. *Thomasmeal*, No 9 : 191-197, (1955).
10. Ghosh, S. K. *J. Ind. Soc. Soil Sci.* 11 (2) : 129-135, (1963).
11. Lask, P *Zn. Pflernahr. Dung*, 67 : 231-239, (1954).
12. Russell, E J. *A Students' book of soils and manures*, XIII Ed. : 237, (1951).
13. Schmitt, L. *Phosphorsaure*, 14 : 227-246, (1954).
14. Volkaerts W. *Phosphorsaure*, 14 : 263-274, (1954).
15. Weihling, Ralph M. *et al. Agri. J. Amer. Soc.* 51 : 87, (1959).

CERTAIN INFINITE AND FINITE INTEGRALS INVOLVING H-FUNCTION AND CONFLUENT HYPERGEOMETRIC FUNCTIONS

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ABSTRACT

In this paper some infinite and finite integrals, involving H-function and confluent hypergeometric functions, have been evaluated.

INTRODUCTION

1. Fox [4, p. 408] introduced the H-function in the form of Mellin-Barnes type integral as

$$(1\cdot1) \qquad \frac{1}{2\pi i}\int_T \frac{\prod_{j=1}^{m}\Gamma(b_j - \beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}\, x^s\, ds,$$

where x is not equal to zero and empty product is interpreted as unity; p, q, m and n are integers satisfying $1 \leqslant m \leqslant q$, $o \leqslant n \leqslant p$; α_j $(j=1,\ldots,p)$, β_j $(j=1,\ldots,q)$ are positive numbers and $a_j (j=1,\ldots,p)$ b_j $(j=1,\ldots,q)$ are complex numbers such that no pole of $\Gamma(b_h - \beta_h s)$, $(h=1,\ldots,m)$ concides with any pole of $\Gamma(1-a_i+\alpha_i s)$ $(i=1,\ldots,n)$, *i.e.*

$$(1\cdot2) \qquad \alpha_i(b_h+\nu) \neq (a_i - n - 1)\,\beta_h$$

$$(\nu, \eta = o, 1, \ldots\ldots; h = 1,\ldots,m; i = 1,\ldots,n)$$

Further the contour T runs from $\sigma - i\infty$ to $\sigma + i\infty$ such that the points:

$$(1\cdot3) \qquad s = \frac{b_h+\nu}{\beta_h} \qquad (h=1,\ldots,m;\ \nu = o, 1, \ldots\ldots),$$

which are poles of $\Gamma(b_h - \beta_h s)$ $(h=1,\ldots,m)$ lie on the right and the points:

$$(1\cdot4) \qquad s = \frac{a_i - \eta - 1}{\alpha_i}\ (i=1,\ldots,n;\ \eta = o, 1, \ldots\ldots),$$

which are the poles of $\Gamma(1-a_i+\alpha_i s)$ $(i=1,\ldots,m)$ lie to the left of T. Such a contour is possible on account of (1·2). These assumptions of the H-function will be adhered to throughout this paper.

Recently Gupta and Jain[5] have denoted (1·1) symbolically by

(1·5) $$H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}(a_1, \alpha_1), \ldots, (a_p, \alpha_p)\\ (b_1, \beta_1), \ldots, (b_q, \beta_q)\end{matrix}\right]$$

and in a more compact form by

(1·6) $$H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right]$$

where $\{(f_r, \gamma_r)\}$ stands for set of the parameters $(f_1, \gamma_1), \ldots, (f_r, \gamma_r)$.

According to Braaksma [3, p. 278]

$$H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right] = O\left(|x|^{\alpha}\right) \text{ for small } x,$$

where $\sum_1^p \alpha_j - \sum_1^q \beta_j \leqslant 0$ and $\alpha = \min Re\left(\frac{b_h}{\beta_h}\right) (h = 1, \ldots, m)$

and

$$H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right] = O\left(|x|^{\beta}\right) \text{ for large } x,$$

where $\sum_1^p \alpha_j - \sum_1^q \beta_j < 0;\ \sum_1^n \alpha_j - \sum_{n+1}^p \alpha_j + \sum_1^n \beta_j - \sum_{n+1}^q \beta_j \equiv \lambda > 0,\ |\arg z| < \frac{1}{2}\lambda\pi$

and $$\beta = \max Re\left(\frac{a_i - 1}{\alpha_i}\right) (i = 1, \ldots, n).$$

The object of this paper is to evaluate some infinite and finite integrals, involving product of the H-function, and confluent hypergeometric functions. As the H-function is a very general function, we get, on specializing the parameters, many cases, some of which are known and others are believed to be new.

2. In this section we state the properties and results, given by Gupta and Jain[5] which will be used in our present work.

The H-function is symmetric in the pairs $(a_1, \alpha_1), \ldots, (a_n, \alpha_n)$ likewise $(a_{n+1}, \alpha_{n+1}), \ldots, (a_p, \alpha_p)$; in $(b_1, \beta_1), \ldots, (b_m, \beta_m)$ and in $(b_{m+1}, \beta_{m+1}), \ldots, (b_q, \beta_q)$.

If one of (a_j, α_j) $(j = 1, \ldots, n)$ is the same as one of (b_h, β_h) $(h = m + 1, \ldots, q)$ or one of (b_h, β_h) $(h = 1, \ldots, m)$ is the same as one of (a_j, α_j) $(j = n + 1, \ldots, p)$ then the H-function reduces to one of lower order *i. e.* each of p, q and n or m decreases by unity.

(2·1) $$x^{\sigma} H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right] \equiv H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}\{(a_p + \sigma\alpha_p, \alpha_p)\}\\ \{(b_q + \sigma\beta_q, \beta_q)\}\end{matrix}\right]$$

(2·2) $$H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}\{(a_p, \alpha_p)\}\\ \{(b_q, \beta_q)\}\end{matrix}\right] \equiv H_{q,p}^{n,m}\left[\frac{1}{x} \,\middle|\, \begin{matrix}\{(1 - b_q, \beta_q)\}\\ \{(1 - a_p, \alpha_p)\}\end{matrix}\right]$$

(2·3) $$H^{m,\,n}_{p,\,q}\left[x\,\middle|\,\begin{matrix}\{(a_p,\,\alpha_p)\}\\ \{(b_q,\,\beta_q)\}\end{matrix}\right]\equiv c\,H^{m,\,n}_{p,\,q}\left[x^c\,\middle|\,\begin{matrix}\{(a_p,\,c\alpha_p)\}\\ \{(b_q,\,c\beta_q)\}\end{matrix}\right],$$

where $c > 0$.

(2·4) $$H^{m,\,n}_{p,\,q}\left[x\,\middle|\,\begin{matrix}\{(a_p,\,\alpha_p)\}\\ \{b_q,\,\beta_q)\}\end{matrix}\right]\equiv(2\pi)^{(1-t)\,(m+n-\frac{1}{2}\,p-\frac{1}{2}q)}\,t^{\mu}$$

$$\times\,H^{t\,m,\,t\,n}_{t\,p,\,t\,q}\left[x\,t^{\tau}\,\middle|\,\begin{matrix}\{(\triangle(t,\,a_p),\,\alpha_p/t)\}\\ \{(\triangle(t,\,b_q),\,\beta_q/t)\}\end{matrix}\right],$$

where μ, τ and $\{(\triangle(t,f_r),\,\gamma_r)\}$ stand for the quantities

$$\sum_1^q b_j-\sum_1^p a_j+\tfrac{1}{2}\,p-\tfrac{1}{2}q\,;\quad \sum_1^p \alpha_j-\sum_1^q \beta_j \text{ and } \left\{\left(\frac{f_r}{t},\,\gamma_r\right)\right\},\ \left\{\left(\frac{f_r+1}{t},\gamma_r\right)\right\},\ldots,$$

$\left\{\left(\frac{f_r+t-1}{t},\,\gamma_r\right)\right\}$ respectively, provided that t is a positive integer greater than 2.

(2·5) $$H^{m,\,n}_{p,\,q}\left[x\,\middle|\,\begin{matrix}\{(a_p,\,1)\}\\ \{(b_q,\,1)\}\end{matrix}\right]\equiv G^{m,\,n}_{p,\,q}\left[x\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right]$$

(2·6) $$H^{p,\,1}_{q+1,\,p}\left[x\,\middle|\,\begin{matrix}(1,\,1),\,\{(b_q,\,1\}\\ \{(a_p,\,1)\}\end{matrix}\right]\equiv E\,(a_1,\ldots,a_p\,;\,b_1,\ldots,b_q\,;\,x).$$

(2·7) $$H^{2,\,0}_{0,\,2}\left[\tfrac{1}{4}\,x^2\,\middle|\,(\tfrac{1}{2}\,l-\tfrac{1}{2}\,\nu,\,1),\,(\tfrac{1}{2}\,l+\tfrac{1}{2}\,\nu,\,1)\right]\equiv 2^{1-l}\,x^l\,K_\nu\,(x),$$

where $K_\nu\,(x)$ is a modified Bessel function.

(2·8) $$H^{2,\,0}_{1,\,2}\left[x\,\middle|\,\begin{matrix}(l-\lambda+1;\,1)\\ (l+\mu+\frac{1}{2},\,1),\,(l-\mu+\frac{1}{2},\,1)\end{matrix}\right]\equiv x^l\,e^{\frac{1}{2}x}\,W_{\lambda,\mu}\,(x),$$

where $W_{\lambda,\mu}\,(x)$ is a Whittaker function.

(2·9) $$H^{1,\,p}_{p,\,q+1}\left[x\,\middle|\,\begin{matrix}\{(1-a_p,\,\alpha_p)\}\\ (0,1),\,\{(1-b_q,\,\beta_q)\}\end{matrix}\right]\equiv\sum_{r=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+\alpha_j r)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j r)}\cdot\frac{(-x)^r}{r\,!};$$

The above series was studied by Wright [7, p. 287] and has been called as Wright's generalised hypergeometric function and is denoted by the symbol

$$_p\psi_q\left[\begin{matrix}\{(a_p,\,\alpha_p)\}\\ \{(b_q,\,\beta_q)\}\end{matrix};\,-x\right].$$

(2·10) $$H^{1,\,0}_{0,\,2}\left[x\,\middle|\,(0,\,1),\,(-\nu,\,\mu)\right]\equiv\sum_{r=0}^{\infty}\frac{(-x)^r}{r\,!\,\Gamma(1+\nu+\mu r)}\equiv J^{\mu}{}_{\nu}(x),$$

where $J^{\mu}_{\nu}(x)$ is Maitland's generalised Bessel function [8, p. 257]

$$(2{\cdot}11) \qquad H^{1,1}_{1,1}\left[x \left| \begin{matrix} (1-\nu, 1) \\ (0, 1) \end{matrix} \right. \right] \equiv \Gamma(\nu)\,(1+x)^{-\nu} \equiv \Gamma(\nu)\, {}_1F_0(\nu, -x).$$

3. In our discussion, because of large number of parameters the notation $(\Delta(\delta, a \pm b), 1)$ will stand for $(\Delta(\delta, a+b), 1)$, $(\Delta(\delta, a-b), 1)$.

The following integrals will be evaluated in this section :

$$(3{\cdot}1) \quad \int_0^\infty x^{\rho-1} \sin(c\, x^{\frac{1}{2}})\, e^{-\frac{x}{2}}\, W_{k,\mu}(x)\, H^{m,n}_{p,q}\left[z\, x^{\delta/t} \left| \begin{matrix} \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix} \right. \right] dx$$

$$= (2\pi)^{\frac{1}{2}(1-\delta)+(1-t)(m+n-\frac{1}{2}p-\frac{1}{2}q)}\; t^{\sum_1^q b_j - \sum_1^p a_j + \frac{1}{2}p - \frac{1}{2}q + 1}\; c\, \delta^{\rho+k}$$

$$\times \sum_{r=0}^{\infty} \frac{(-c^2/4)^r\, \delta^r}{r!\,(\frac{3}{2})_r}$$

$$\times H^{tm,\, tn+2\delta}_{tp+2\delta,\, tq+\delta}\left[(z^{t\tau})^t\, \delta^{\delta} \left| \begin{matrix} (\Delta(\delta, -\rho-r\pm\mu), 1), \{(\Delta(t, a_p), \alpha_p)\} \\ \{(\Delta(t, b_q), \beta_q)\}, (\Delta(\delta, -\rho-r+k-\frac{1}{2}), 1) \end{matrix} \right. \right]$$

where δ and t are positive integers, $\sum_1^p \alpha_j - \sum_1^q \beta_j \equiv \tau \leqslant 0$,

$$\sum_1^n \alpha_j - \sum_{n+1}^p \alpha_j + \sum_1^m \beta_j - \sum_{m+1}^q \beta_j \equiv \lambda > 0,\ |\arg z| < \tfrac{1}{2}\lambda\pi \text{ and}$$

$$Re\,(\rho + \tfrac{\delta}{t}\, b_h/\beta_h) > |\,Re\,\mu\,| - 1 \ (h = 1, \ldots, m).$$

$$(3{\cdot}2) \qquad \int_0^\infty x^{\rho-1} \cos(c\, x^{1/2})\, e^{-\frac{1}{2}x}\, W_{k,\mu}(x)\, H^{m,n}_{p,q}\left[z\, x^{\delta/t} \left| \begin{matrix} \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix} \right. \right] dx$$

$$= (2\pi)^{\frac{1}{2}(1-\delta)+(1-t)(m+n-\frac{1}{2}p-\frac{1}{2}q)}\; t^{\sum_1^q b_j - \sum_1^p a_j + \frac{1}{2}p - \frac{1}{2}q + 1}\; \delta^{\rho+k-\frac{1}{2}}$$

$$\times \sum_{r=0}^{\infty} \frac{(-c^2/4)^r\, \delta^r}{r!\,(\frac{1}{2})_r}$$

$$\times H^{tm,\, tn+2\delta}_{tp+2\delta,\, tq+\delta}\left[(zt^{\tau})^t\, \delta^{\delta} \left| \begin{matrix} (\Delta(\delta, \frac{1}{2}-\rho-r\pm\mu), 1), \{(\Delta(t, a_p), \alpha_p)\} \\ \{(\Delta(t, b_q), \beta_q)\}, (\Delta(\delta, k-\rho-r), 1) \end{matrix} \right. \right],$$

where δ and t are positive integers, $\sum_1^p \alpha_j - \sum_1^q \beta_j \equiv \tau \leqslant 0$,

$$\sum_1^n \alpha_j - \sum_{n+1}^p \alpha_j + \sum_1^m \beta_j - \sum_{m+1}^q \beta_j \equiv \lambda > 0,\ |\arg z| < \tfrac{1}{2}\lambda\pi \text{ and}$$

$$Re\,(\rho + \delta/t.\ b_h/\beta_h) > |\,Re\,\mu\,| - \tfrac{1}{2}\ (h = 1, \ldots, m).$$

$$(3{\cdot}3) \qquad \int_0^\infty x^{\rho-1} e^{-x/2} J_{\lambda+\nu}(\alpha x^{\frac12}) J_{\lambda-\nu}(\alpha x^{\frac12})\, W_{k,\mu}(x)$$

$$\times H\,{}^{m,\,n}_{p,\,q}\left[z\, x^{\delta/t} \,\middle|\, \begin{matrix} \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix} \right] dx$$

$$= (2\pi)^{\frac12(1-\delta)+(1-t)(m+n-\frac12 p-\frac12 q)}\; t^{\sum_1^q b_j - \sum_1^p a_j + \frac12 p - \frac12 q + 1}\; \delta^{\rho+k+\lambda-\frac12}\, (\tfrac12 \alpha)^{2\lambda}$$

$$\times \sum_{r=0}^{\infty} \frac{(1+\lambda)_r (\frac12+\lambda)_r}{(1+2\lambda)_r} \cdot \frac{1}{\Gamma(1+\lambda+\nu+r)\,\Gamma(1+\lambda-\nu+r)} \cdot \frac{(-\alpha^2)^r}{r!}$$

$$\times H\,{}^{tm,\; tn+2\delta}_{tp+2\delta,\; tq+\delta}\left[(z t^{\tau})^t \delta^{\delta} \,\middle|\, \begin{matrix} (\Delta(\delta, \frac12-\rho-\lambda-r\pm\mu), 1)\{(\Delta(t, a_p), \alpha_p)\} \\ \{(\Delta(t, b_q), \beta_q)\}, (\Delta(\delta, k-\rho-\lambda-r), 1) \end{matrix} \right],$$

where δ and t are positive integers, $\sum_1^p \alpha_j - \sum_1^q \beta_j \equiv \tau \leqslant 0$,

$\sum_1^n \alpha_j - \sum_{n+1}^p \alpha_j + \sum_1^m B_j - \sum_{m+1}^q \beta_j \equiv \lambda > 0$, $|\arg z| < \frac12 \lambda\pi$ and

$$Re\left(\rho + \lambda + \frac{\delta}{t} \cdot \frac{b_h}{\beta_h} \right) > |Re\, \mu| - \tfrac12 \; (h = 1, \ldots, m).$$

$$(3{\cdot}4) \qquad \int_0^\infty x^{\rho-1} e^{-\frac12(\alpha+\beta)x} M_{k,\mu}(\alpha x)\, W_{\lambda,\nu}(\beta x)\, H\,{}^{m,\,n}_{p,\,q}\left[z x^{\delta/t} \,\middle|\, \begin{matrix} \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix} \right] dx$$

$$= (2\pi)^{\frac12(1-\delta)+(1-t)(m+n-\frac12 p-\frac12 q)}\; t^{\sum_1^q b_j - \sum_1^p a_j + \frac12 p - \frac12 q + 1}\; \alpha^{\mu+1/2}\, \beta^{-\mu-\rho-1/2}$$

$$\times \delta^{\mu+\rho+\lambda} \sum_{r=0}^{\infty} \frac{(\frac12+k+\mu)_r\, (-\alpha/\beta)^r\, \delta^r}{(2\mu+1)_r\, r!}$$

$$\times H\,{}^{tm,\; tn+2\delta}_{tp+2\delta,\; tq+\delta}\left[(z t^{\tau})^t (\delta/\beta)^{\delta} \,\middle|\, \begin{matrix} (\Delta(\delta, -\rho-\mu-r\pm\nu), 1), \{(\Delta(t, a_p)\alpha_p)\} \\ \{(\Delta(t, b_q), \beta_q)\}, (\Delta(\delta, -\rho-\mu+\lambda-\frac12-r), 1) \end{matrix} \right],$$

where δ and t are positive integers, $\sum_1^p \alpha_j - \sum_1^q \beta_j \equiv \tau \leqslant 0$, $Re\,(\alpha) > 0$

$Re\,(\beta) > 0$, $\sum_1^n \alpha_j - \sum_{n+1}^p \alpha_j + \sum_1^m \beta_j - \sum_{n+1}^q \beta_j \equiv \lambda > 0$, $|\arg z| < \frac12 \lambda\pi$ and

$$Re\left(\rho + \mu + \frac{\delta}{t} \cdot \frac{b_h}{\beta_h} \right) > |Re\, \nu| - 1 \; (h = 1, \ldots, m).$$

$$(3{\cdot}5)\quad \int_0^a x^{\beta-1}(a-x)^{\gamma-1}\,{}_1F_1(\alpha\,;\beta\,::x)\,H_{p,q}^{m,n}\left[z(a-x)^{\delta/t}\,\middle|\,\begin{matrix}\{(a_p,\alpha_p)\}\\\{(b_q,\beta_q)\}\end{matrix}\right]dx$$

$$= (2\pi)^{(1-t)(m+n-\frac{1}{2}p-\frac{1}{2}q)}\; t^{\sum_1^q b_j-\sum_1^p a_j+\frac{1}{2}p-\frac{1}{2}q+1}\;\Gamma(\beta)\;\delta^{-\beta}\;a^{\beta+\gamma-1}$$

$$\times\sum_{r=0}^{\infty}\frac{a^r\,\delta^{-r}\,(\alpha)_r}{r!}$$

$$\times\; H_{tp+\delta,\,tq+\delta}^{tm,\,tn+\delta}\left[(zt^{\tau})^t a^{\delta}\,\middle|\,\begin{matrix}(\Delta(\delta,1-\gamma),1),\{(\Delta(t,a_p),\alpha_p)\}\\\{(\Delta(t,b_q),\beta_q)\},(\Delta(\delta,1-\beta-\gamma-r),1)\}\end{matrix}\right],$$

where δ and t are positive integers, $Re\,(\beta)>0$, $\sum_1^p \alpha_j-\sum_1^q \beta_j\equiv\tau\leqslant 0$,

$\sum_1^n \alpha_j-\sum_{n+1}^p \alpha_j+\sum_1^m \beta_j-\sum_{n+1}^q \beta_j\equiv\lambda>0$, $|\arg z|<\frac{1}{2}\lambda\,\pi$ and $Re\left(\gamma+\frac{\delta}{t}\cdot\frac{b_h}{\beta_h}\right)>0$ $(h=1,\ldots,n)$.

Proof: Initially we start with $t=1$ in the integral $(3{\cdot}1)$. Expressing the H-function in its integrand in the form of Mellin-Barnes type of integral $(1{\cdot}1)$ and interchanging the order of integration, which is justifiable under the conditions stated in $(3{\cdot}1)$, we get:

$$(3{\cdot}6)\quad \frac{1}{2\pi i}\int_T \frac{\prod_{j=1}^m\Gamma(b_j-\beta_j s)\prod_{j=1}^n\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^q\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^p\Gamma(a_j-\alpha_j s)}\,z^s\,ds=$$

$$\times\int_0^{\infty} x^{\rho+s\delta-1}\sin(c\,x^{1/2})\,e^{-x/2}\,W_{k,\mu}(x)\,dx.$$

After evaluating the x-integral with the help of the result [2, p. 407 (28)] and using the Gauss' multiplication theorem for Gamma functions [6, p. 26], $(3{\cdot}6)$ reduces to

$$(3{\cdot}7)\quad (2\pi)^{\frac{1}{2}(1-\delta)}\,c\,\delta^{\rho+k}\sum_{r=0}^{\infty}\frac{(-c^2/4)^r\,\delta^r}{r!\,(\frac{3}{2})_r}$$

$$\times\frac{1}{2\pi i}\int_T\frac{\prod_{j=1}^m\Gamma(b_j-\beta_j s)\prod_{j=1}^n\Gamma(1-a_j+\alpha_j s)\prod_{\eta=0}^{\delta-1}\Gamma\left(\frac{1+\mu+\rho+r+\eta}{\delta}+s\right)}{\prod_{j=m+1}^q\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^p\Gamma(a_j-\alpha_j s)\prod_{\eta=0}^{\delta-1}\Gamma\left(\frac{\frac{3}{2}+\rho-k+r+\eta}{\delta}+s\right)}\times$$

$$\times\prod_{\eta=0}^{\delta-1}\Gamma\left(\frac{1-\mu+\rho+r+\eta}{\delta}+s\right)z^s\,\delta^{s\delta}\,ds.$$

Therefore in accordance with the definition $(1{\cdot}1)$ of the H-function, $(3{\cdot}7)$ yields the value of the integral $(3{\cdot}1)$ with $t=1$ as

(3·8) $$(2\pi)^{\frac{1}{2}(1-\delta)}\, c\,\delta^{\rho+k} \sum_{r=0}^{\infty} \frac{(-c^2/4)^r\, \delta^r}{r!\,(\frac{3}{2})_r}$$

$$\times H^{m,\, n+2\delta}_{p+2\delta,\, q+\delta}\left[z\delta^{\delta} \,\middle|\, \begin{matrix} (\Delta(\delta, -\rho-r\pm\mu), 1), \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\}, (\Delta(\delta, -\rho-r+k-\frac{1}{2}), 1) \end{matrix} \right],$$

In (2·4) replacing x by $(z\, x^{\delta/t})$ and using (2·3) we get :

(3·9) $$H^{m,\,n}_{p,\,q}\left[z\,x^{\delta/t} \,\middle|\, \begin{matrix} \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix} \right] \equiv (2\pi)^{(1-t)\,(m+n-\frac{1}{2}p-\frac{1}{2}q}\; t^{\sum_1^q b_j - \sum_1^p a_j + \frac{1}{2}p - \frac{1}{2}q + 1}$$

$$\times H^{tm,\, tn}_{tp,\, tq}\left[(zt^{\tau})^t\, x^{\delta} \,\middle|\, \begin{matrix} \{(\Delta(t, a_p), \alpha_p)\} \\ \{(\Delta(t, b_q), \beta_q)\} \end{matrix} \right],$$

where τ stands for $\left(\sum_1^p \alpha_j - \sum_1^q \beta_j\right)$.

By virtue of (3·9) the integral (3·1) can easily be deduced from (3·8) on making proper substitutions.

Following the same procedure as above and using the results [2, p. 407(30)], [2, p. 409(37)], [2, p. 410(43)] and [2, p. 401(1)] respectively, the integrals (3·2), (3·3), (3·4) and (3·5) can be easily established.

4. *Particular Cases* : In view of the section 2 by setting the parameters suitably, the H-function and confluent hypergeometric functions, involved in the integrals, evaluated above in the section 3, will yield many simple functions and thus so many integrals may be obtained as their special cases. However, few interesting cases, some of which give generalisation of certain known results, are mentioned here.

Taking $\alpha_j = \beta_j = 1$ $(j = 1, \ldots, p;\; h = 1, \ldots, q)$ in (3·1), we come to a known case, recently obtained by Bajpai[1] as :

(4·1) $$\int_0^{\infty} x^{\rho-1} \sin(c\,x^{1/2})\, e^{-\frac{1}{2}x}\, W_{k,\mu}(x)\, G^{m,\,n}_{p,\,q}\left[z\,x^{\delta/t} \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \right] dx$$

$$= (2\pi)^{(1-t)\,(m+n-\frac{1}{2}p-\frac{1}{2}q)+\frac{1}{2}-\frac{1}{2}\delta}\; t^{\sum_1^q b_j - \sum_1^p a_j + \frac{1}{2}p - \frac{1}{2}q + 1}\, c\,\delta^{\rho+k}$$

$$\times \sum_{r=0}^{\infty} \frac{(-c^2/4)^r\, \delta^r}{r!\,(\frac{3}{2})_r}$$

$$\times G^{tm,\, tn+2\delta}_{tp+2\delta,\, tq+\delta}\left[(zt^{p-q})^t \,\middle|\, \begin{matrix} \Delta(\delta, -\rho-r\pm\mu), \Delta(t, a_1), \ldots, \Delta(t, a_p) \\ \Delta(t, b_1), \ldots, \Delta(t, b_q), \Delta(\delta, -\rho-r+k-\frac{1}{2}) \end{matrix} \right],$$

where δ and t are positive integers, $p + q < 2\,(m + n)$, $|\arg z| < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$ and $Re\,(\rho+\delta/t\; b_j) < |Re\,\mu| - 1$ $(j = 1, \ldots, m)$.

In (3·1) replacing m, n, p, q by $p, 1, q+1, p$ respectively and setting the other parameters suitably in view of (2·6) we obtain

$$(4\cdot 2)\qquad \int_0^\infty x^{\rho-1}\sin(cx^{\frac{1}{2}})\,e^{-\frac{1}{2}x}\,W_{k,\mu}(x)\,E(a_1,\ldots,a_p\,;\,b_1,\ldots,b_q\,;\,zx^{\delta/t})\,dx$$

$$=(2\pi)^{\frac{1}{2}(1-\delta)+\frac{1}{2}(1-t)(pn\,q+1)}\,t^{\sum_1^p a_j-\sum_1^q b_j-\frac{1}{2}p+\frac{1}{2}q+\frac{1}{2}}\,c\delta^{\rho+k}$$

$$\times\sum_{r=0}^{\infty}\frac{\left(-\frac{c^2}{4}\right)^r\delta^r}{r!\,(\frac{3}{2})_r}$$

$$\times\,G^{tp,\,t+2\delta}_{tq+t+2\delta,\,tp+\delta}\left[(zt^{\tau})^t\,\delta^\delta\,\middle|\,\begin{matrix}\Delta(\delta,-\rho-r\pm\mu),\,\Delta(t,1),\,\Delta(t,b_1),\ldots,\Delta(t,b_q)\\ \Delta(t,a_1),\ldots,\Delta(t,a_p),\,\Delta(\delta,-\rho-r+k-\frac{1}{2})\end{matrix}\right]$$

provided that t and δ are positive integers, $q-p+1\equiv\tau\leqslant 0$, $p-q+1\equiv\lambda>0$, $|\arg z|<\frac{1}{2}\lambda\pi$ and $Re(\rho+\delta/t\,a_j)>|Re(\mu)|-1$ $(j=1,\ldots,p)$.

Putting $m=q=2$, $n=p=0$, $b_1=\frac{1}{2}l-\frac{1}{2}\nu$, $b_2=\frac{1}{2}l+\frac{1}{2}\nu$ and $\beta_1=\beta_2=1$, the integral (3·1), in view of (2·7), reduces to

$$(4\cdot 3)\qquad \int_0^\infty x^{\rho+\frac{l\delta}{2t}-1}\sin(cx^{1/2})\,e^{-\frac{1}{2}x}\,W_{k,\mu}(x)\,K_\nu(2\sqrt{z}\,x^{\delta/2t})\,dx$$

$$=(2\pi)^{\frac{3}{2}-\frac{1}{2}\delta-t}\,\frac{c}{2}\,z^{-l/2}\,t^l\,\delta^{\rho+k}\sum_{r=0}^{\infty}\frac{\delta^r\left(-\frac{c^2}{4}\right)^r}{r!\,(\frac{3}{2})_r}$$

$$\times\,G^{2t,\,2\delta}_{2\delta,\,2t+\delta}\left[(zt^{-2})^t\,\delta^\delta\,\middle|\,\begin{matrix}\Delta(\delta,-\rho-r\pm\mu)\\ \Delta(\delta,\frac{1}{2}l\pm\frac{\nu}{2}),\,\Delta(\delta,-\rho-r+k-\frac{1}{2})\end{matrix}\right]$$

where δ and t are positive integers, $|\arg z|<\pi$ and

$$Re[\rho+\delta/t\,(\tfrac{1}{2}t\pm\tfrac{1}{2}\nu)]>|Re\,\mu|-1.$$

Taking $m=q=2$, $n=0$, $p=1$, $a_1=l-\lambda+1$, $b_1=\frac{1}{2}+l+\nu$, $b_2=\frac{1}{2}+l-\nu$ and $\alpha_1=\beta_1=\beta_2=1$ in (3·1) and using (2·8) we get

$$(4\cdot 4)\qquad \int_0^\infty e^{-\frac{1}{2}x-\frac{1}{2}zx^{\delta/t}}\,x^{\rho+\frac{l\delta}{t}-1}\sin(cx^{1/2})\,W_{k,\mu}(x)\,W_{\lambda,\nu}(zx^{\delta/t})\,dx$$

$$=(2\pi)^{\frac{1}{2}(2-\delta-t)}\,cz^{-l}\,t^{\frac{1}{2}+l+\lambda}\,\delta^{\rho+k}\sum_{r=0}^{\infty}\frac{\left(-\frac{c^2}{4}\right)^r\delta^r}{r!\,(\frac{3}{2})_r}$$

$$\times\,G^{2t,\,2\delta}_{t+2\delta,\,2t+\delta}\left[(zt^{-1})^t\,\delta^\delta\,\middle|\,\begin{matrix}\Delta(\delta,-\rho-r\pm\mu),\,\Delta(t,l-\lambda+1)\\ \Delta(t,\frac{1}{2}+l\pm\nu),\,\Delta(\delta,-\rho-r+k-\frac{1}{2})\end{matrix}\right],$$

provided that t and δ are positive integers, $|\arg z|<\frac{1}{2}\pi$ and $Re[\rho+\delta/t(\frac{1}{2}+l\pm\nu)]>|Re\,\mu|-1.$

In (3·1) replacing m, n, q by $t, p, q+1$ respectively and choosing the other parameters suitably in view of (2·9), we obtain

(4·5) $$\int_0^\infty x^{\rho-1} e^{-\frac{1}{2}x} \sin (cx^{\frac{1}{2}})\, W_{k,\mu}(x)\, {}_p\psi_q \left[\begin{matrix}\{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\}\end{matrix} ; - zx^{\delta/t}\right] dx$$

$$= (2\pi)^{\frac{1}{2}(1-t)(1+\lambda-\tau)+\frac{1}{2}(1-\delta)}\, t^{\sum_1^p a_j - \sum_1^q b_j+\frac{1}{2}q - \frac{1}{2}p+\frac{1}{2}}\, c\delta^{\rho+k}$$

$$\times \sum_{r=0}^\infty \frac{\left(-\frac{c^2}{4}\right)^r \delta^r}{r!\,(\frac{3}{2})_r}$$

$$\times H^{t,\, tp+2\delta}_{tp+2\delta,\, tq+t+\delta}\left[(zt^{\tau})^t \delta^\delta \left| \begin{matrix}(\Delta(\delta,-\rho-r\pm\mu),1),\ \{(\Delta(t,1-a_p),\alpha_p)\} \\ (\Delta(t,0),1)\ \{(\Delta(t,1-b_q),\beta_q)\},\ (\Delta(\delta,-\rho-r+k-\frac{1}{2}),\ 1)\end{matrix}\right.\right]$$

where δ and t are positive integers, $\sum_1^p \alpha_j - \sum_1^q \beta_j - 1 \equiv \tau \leqslant 0$, $\sum_1^p \alpha_j - \sum_1^q \beta_j+1\equiv\lambda>0$,

$|\arg z| < \frac{1}{2}\lambda\pi$ and $Re(\rho) > |Re\,\mu| - 1$.

In (3·1) setting $m = 1$, $n = p = 0$, $q = 2$, $b_1 = 0$, $b_2 = -\nu$, $\beta_1 = 1$ and $\beta_2 = \mu'$, it, by virtue of (2·10), reduces to

(4·6) $$\int_0^\infty x^{\rho-1} e^{-\frac{1}{2}x} \sin (cx^{\frac{1}{2}})\, W_{k,\mu}(x)\, J_\nu^{\mu'}(zx^{\delta/t})\, dx$$

$$= (2\pi)^{\frac{1}{2}(1-\delta)}\, t^{-\nu}\, c\delta^{\rho+k} \sum_{r=0}^\infty \frac{(-c^2/4)^r \delta^r}{r!\,(\frac{3}{2})_r}$$

$$\times H^{t,\, 2\delta}_{2\delta,\, 2t+\delta}\left[(zt^{-\mu-1})^t \delta^\delta \left| \begin{matrix}\Delta(\delta, -\rho-r\pm\mu), 1) \\ \Delta(t,0), 1), (\Delta(t,-\nu), \mu'), (\Delta(\delta,-\rho-r+k-\frac{1}{2}),1)\end{matrix}\right.\right]$$

provided that δ and t are positive integers, $-1 \leqslant \mu < 1$, $|\arg z| < \frac{1}{2}(1-\mu)\pi$ and $Re(\rho) > |Re\,\mu| - 1$.

With $m = n = p = q = 1$, $a_1 = 1-\nu$, $b_1 = 0$, $\alpha_1 = \beta_1 = 1$ in (3·1) and using (2·11), we get

(4·7) $$\int_0^\infty x^{\rho-1} e^{-\frac{1}{2}x} \sin (c\, x^{1/2})\, W_{k,\mu}(x)\, (1+z\, x^{\delta/t})^{-\nu}\, dx$$

$$= (2\pi)^{3/2-\frac{1}{2}\delta-t}\, t^\nu\, c\delta^{\rho+k} \frac{1}{\Gamma(\nu)} \sum_{r=0}^\infty \frac{(-c^2/4)^r \delta^r}{r!\,(\frac{3}{2})_r}$$

$$\times G^{t,\, t+2\delta}_{t+2\delta,\, t+\delta}\left[z^t \delta^\delta \left| \begin{matrix}\Delta(\delta, -\rho-r\pm\mu),\ \Delta(t,\ 1-\nu) \\ \Delta(t,\ 0),\ \Delta(\delta,\ -\rho-r+k-\frac{1}{2})\end{matrix}\right.\right],$$

where δ and t are positive integers, $|\arg z| < \pi$ and $Re(\rho) > |Re\,\mu| - 1$.

Proceeding on similar lines as above and using the results of the section 2, the integrals (3·2) to (3·5) will also yield many integrals as their particular cases.

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REFERENCES

1. Bajpai, S. D. Some integrals involving confluent hypergeometric functions and Meijer's G functions. *To appear, in Mathematica Japonicae.*
2. Bateman Project. Tables of integrals-transforms, Vol II. *McGraw Gill,* (1954).
3. Braaksma, B L. J. Asymptotic expansions and analytic continuations for a class of Barnes-integrals, *Compos. Math.* 15 : 239-341, (1963).
4. Fox, C. The G and H-functions as symmetrical Fourier kernels. *Trans. Amer. Math. Soc.* 98 : 395-429, (1961).
5. Gupta K C. and Jain, U. C. The H-function—II. *To appear in Proc. Nat. Acad. Sci., India.*
6. Rainville, E. D. *Special functions, Macmillan, New York,* (1960).
7. Wright, E. M. The asymptotic expansion of the generalised hypergeometric function. *Jour. Lond Math. Soc.* 10 : 286-293, (1953).
8. Wright, E. M. The asymptotic expansion of the generalised Bessel function. *Proc. Lond. Math. Soc.*, 38 : 257-270, (1935).

STUDIES IN THE RELEASE OF FIXED AND ADSORBED POTASSIUM IN THE ALKALI SOIL PROFILE

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ABSTRACT

Fixed Potassium is present in all the horizons of the alkali soil profile which is easily released by leaching with normal ammonium acetate and ammonium chloride solutions, and not by normal sodium chloride, N/10 hydrochloric acid or even by normal hydrochloric acid. This shows that the fixed K and NH_4 can be replaced by each other and not by any other ions.

The ability of soil to adsorb and hold potassium is of great importance because it serves to decrease leaching and provides for a more continuous supply of available potash. The amount so fixed is usually directly proportional to the amount of colloidal matter in the soil, being greatest in the heavy soils and least in the light or sandy soils. Fixed potassium moves slowly, if at all, in the soil. The first detailed study of this fixation was that of Volk[1]. His results pointed out that (*a*) drying the soil was extremely important in the process, (*b*) the clay size fraction was primarily responsible but the quality of clay was also important since a kaoline clay sample and two laterite soils with large amount of clay size fraction did not fix potassium, (*c*) hydrochloric acid treatment of a soil decreased the fixing power whereas treatment with Na_2CO_3 or $Ca(OH)_2$ increased it, (*d*) long continued K-fertilization resulted in decreasing the K–fixing power and increasing the muscovite-like component of the clay size fraction and (*e*) the greater the amount of potash added, the greater was the amount fixed upon drying but the smaller was the percentage fixed.

The next important contribution was that of Chaminade[2] who related the fixation to the exchange capacity of the soil. Truog and Jones[3] and Kolodny[4] indicated that the exchangeable ions were the seat of the fixation process. Page and Baver[5] put forward 'the lattice hole theory' for the fixation process. Although most of the early work indicated that appreciable cationic fixation occurred only on drying, it soon became clear that fixation takes place in moist condition also. Stanford and Pierre[6], Stanford[7], Sears[8], Chaminade[2] and Allway and Pierre[9] found large amounts of potassium fixation under moist condition.

A considerable reaearch work has been carried on the release of the fixed potassium in the soil and minerals. Hoagland and Martin[10] found that fixed potassium cannot be released even by leaching with 0·2 N acid. Blume and Purvis[11] reported that fixation of K is highly reversible and that potassium in fixed state is released and again fixed in a comparatively short time. Breazeale and Magistad[12] found that alternate freezing and thawing and prolonged treatment on a steam bath with 0·05 N HCl released no potassium.

The present investigations were undertaken with a view to study the release of fixed and adsorbed potassium in alkali soil profile and the influence of various

extracting electrolytes like NH_4–AC, NH_4Cl, NaCl and HCl upon the release of fixed potassium.

EXPERIMENTAL PROCEDURE

Soil profile for the present investigations was collected from a village situated at a distance of 18 miles from Allahabad. 2·0 gm portions of soil from each layer of the profile were taken and 250 c.c. of distilled water was added and stirred for one hour. It was filtered and then washed with 40% ethanol. Soil freed from soluble salts was transferred to a 400 c.c. beaker. 250 c.c. of neutral normal solutions of NH_4–AC, NH_4Cl, NaCl, HCl, and 0·1 N HCl were added, stirred and kept overnight. Next day, the samples were leached with the above mentioned solutions in order to collect 1 litre of the extract which was analysed for K, Ca, Mg and R_2O_3. Potassium was estimated by sodium cobaltinitrite method[13]. Calcium and magnesium were estimated by precipitating them as calcium oxalate and magnesium ammonium phosphate respectively, as given in C. S. Piper's book[13]. Sesquioxide was estimated by ammonia precipitation[14].

The value of fixed potassium was obtained by subtracting the amount of K released by NaCl from the amount of K released by neutral N NH_4 Cl and NH_4–AC. The amount of fixed potassium released was calculated in lbs. per acre foot on the basis of the weight of soil = 4,000,000 lbs. per acre foot (U. S. Deptt. Agri. Handbook No. 60, 1954).

In order to find out net negative charge, two sets of ammonium saturated samples were subsequently leached with neutral N KCl and neutral N NaCl in order to obtain 1 litre extract. The amount of ammonium ion released was determined in an aliquot portion of the extract with ignited MgO. The amount of NH_4 ion released by Na ions was taken to be equivalent to the exchange capacity of the soil in m. e./100 gm. soil and that released by N KCl was taken as gross negative charge on the soil colloids. The difference to the amount of NH_4 released by K and Na ions represented the net negative charge occupied by the fixed potassium ions.

RESULTS

Description of the profile and the analytical results are recorded below :

DESCRIPTION OF THE PROFILE

Date of collection	..	20th March, 1962.
State	..	Uttar Pradesh
Country	..	India.
Soil type		Alkali
Geographical land-scape	..	About two and a half miles from Babuganj Bus station, (Phulpur), Allahabad.
Geology	..	Alluvial origin.
Microrelief	..	Plain field where nothing was grown due to alkalinity.
Condition and culture	..	Whitish patches all over the area. Few patches of grass like Cyanodon dactylon were observed. Unfit for cultivation due to alkalinity.

TABLE 1

Chemical properties and mechanical composition of the alkali soil profile

	DEPTH IN INCHES						
	0–6	6–18	18–30	30–42	42–54	54–66	66–78
Loss on ignition %	2·65	2·82	3·04	4·80	5·10	5·30	3·62
HCl insoluble %	84·32	83·40	81·25	71·95	68·05	75·86	79·00
Sesquioxide %	7·15	8·12	10·34	30·25	11·30	11·82	10·65
Fe_2O_3 %	4·12	4·45	4·62	6·50	4·90	5·32	5·82
CaO %	1·26	1·24	1·28	5·45	11·20	5·25	4·96
MgO %	1·23	1·33	1·45	1·65	1·80	0·60	0·58
K_2O %	0·79	1·15	1·30	1·52	1·86	0·80	0·72
P_2O_5 %	0·135	0·145	0·168	0·170	0·186	0·138	0·126
$CaCO_3$ equivalent	2·48	1·75	2·20	10·50	17·00	10·35	4·42
Carbon %	0·2108	0·1946	0·1805	0·1802	0·1778	0·1776	0·1568
Nitrogen %	0·040	0·040	0·034	0·030	0·030	0·026	0·025
pH	9·3	9·5	9·4	9·2	9·1	9·0	8·8
Electrical conductivity m. mohs/cm. at 25°C.	11·85	12·20	11·75	11·02	10·32	9·02	8·15
Cation exchange capacity m.e. %	14·54	15·65	17·30	19·65	15·52	12·23	11·45
Exch. Ca. m. e. %	4·62	5·23	6·65	8·05	7·15	5·25	4·85
Exch. Mg. m. e. %	1·70	1·55	1·95	2·75	2·62	2·46	2·40
Exch. K. m. e. %	0·20	0·26	2·28	0·32	0·36	0·30	0·34
Exch. Na. m. e. %	7·52	7·85	8·05	8·42	4·26	2·65	2·42
			Mechanical composition				
Coarse sand % (2·0–0·2 mm.)	8·45	2·62	2·40	1·65	1·70	9·46	17·54
Fine sand % (0·2–0·02 mm.)	45-50	40·00	25·24	21·45	22·45	38·54	48-62
Silt % (0·02–0·002 mm.)	32·50	40·20	47·52	49·82	48·32	26·62	20·42
Clay % (less than 0·002 mm.)	13·10	15·20	21·62	24·45	23·42	23·35	8·82

TABLE 2

Release of fixe l and adsorbed potassium in calcareous alkaline soil profile by different extracting solutions (m. e./100 gm./soil, oven dry)

	DEPTH IN INCHES						
	0 – 6	6 – 18	18 – 30	30 – 42	42 – 54	54 – 66	66 – 78
N NH_4–AC							
K^+	12·65	14·65	20·65	30·42	35·56	22·54	26·42
Ca^{++}	26·24	28·05	35·26	46·52	55·24	37·23	43·24
Mg^{++}	4·95	5·06	5·48	8·25	9·16	6·15	6·75
N NH_4Cl							
K^+	12·26	14·02	20·06	29·65	34·02	22·00	25·90
Ca^{++}	28·62	30·62	37·54	48·42	58·65	39·40	45·06
Mg^{++}	5·15	5·30	5·70	8·76	9·80	6·62	8·26
N HCl							
K^+	2·06	2·00	2·21	2·40	2·62	1·80	2·12
Ca^{++}	105·60	109·00	128·60	156·50	188·50	130·82	134·24
Mg^{++}	18·80	20·60	35·80	62·62	85·40	28·45	21-62
R_2O_3	1·32	1·60	2·00	2·35	2·56	1·72	1·82
0·1N HCl							
K^+	0·29	0·30	0·46	0·73	0·82	0·54	0·66
Ca^{++}	36·20	40·00	50·56	58·25	71-42	52·45	54·65
Mg^{++}	5·52	5·80	6·02	11·05	11·86	10·14	10·50
N NaCl							
K^+	0·18	0·16	0·26	0·65	0·70	0·48	0·54
Ca^{++}	24·62	26·46	28·26	24·85	27·42	19·50	19·56
Mg^{++}	1·62	1·75	2·80	4·95	6·01	3·54	4·16

TABLE 3

Adsorbed Potassium in calcareous alkaline soil profile (m. e./100 gm. soil, oven dry)

Depth in inches	Normal ammonium acetate	Normal ammonium chloride
0–6	12·65	12·62
6–18	14·65	14·02
18-30	20·65	20·06
30-42	30·42	29·65
42-54	35·56	34·02
54–66	22·54	22·00
66-78	26·42	25·90

TABLE 4

Exchangeable potassium in calcareous alkaline soil Profile (lbs. per acre foot soil)

Depth in inches	Exchangeable potassium (lbs.)
0–6	312·4
6–18	406·4
18–30	437·6
30–42	500·4
42-54	562·8
54–66	468·8
66-78	531·6

TABLE 5

Fixed potassium in calcareous alkaline soil profile (m. e./100 gm. soil, oven dry)

Depth in inches	Normal ammonium acetate	Normal ammonium chloride
0–6	12·47	12·08
6–18	14·49	13·86
18–30	20·39	19·80
30–42	29·77	29·00
42-54	34·86	33·32
54-66	22·06	21·52
66-78	25·88	25·36

TABLE 6

Fixed potassium in calcareous alkaline soil profile (lbs./acre foot of soil)

Depth in inches	Normal ammonium acetate	Normal ammonium chloride
0–6	19500	18885
6-18	22660	21672
18-30	31884	30964
30–42	46552	45348
42–54	54512	52104
54–66	34596	33652
66–78	40472	39656

TABLE 7

Release of fixed potassium not released by hydrogen and sodium ions by leaching with neutral normal ammonium chloride (m. e./100 gm. soil, oven dry)

Depth in inches	0·1N hydrochloric acid	Normal sodium chloride
0–6	11·85	11·90
6-18	13·64	13·76
18–30	19·52	19·70
30–42	28·88	28·95
42–54	33·17	33·28
54–66	21·44	21·49
66-78	25·22	25·32

TABLE 8

Release of fixed and adsorbed ammonium ions in calcareous alkaline soil profile saturated with ammonium ions by ammonium acetate (m. e./100 gm. soil, oven dry)

Determination of gross and net negative charges

Depth in inches	Gross	Exchange complex	Net
0–6	20·45	7·93	12·52
6–18	25·06	10·44	14·62
18–30	33·65	13·03	20·62
30–42	48·62	18·24	30·38
42–54	48·06	31·64	34·42
54–66	28·62	6·36	22·26
66–78	32·24	5·84	26·40

DISTRIBUTION OF EXCH. 'K' IN ALKALI SOIL PROFILE IN Lbs. PER ACRE FOOT OF SOIL

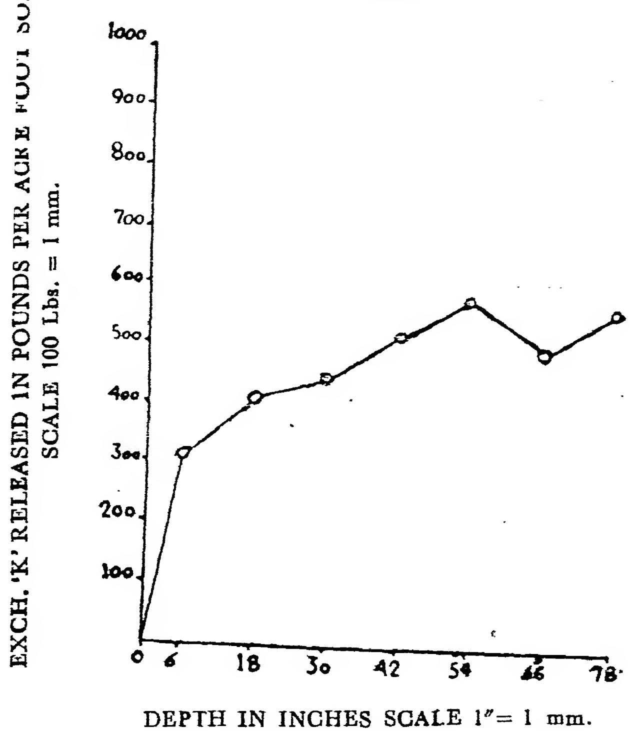

FIG. 1

RELEASE OF FIXED 'K' IN ALKALI SOIL PROFILE BY N NH_4 AC

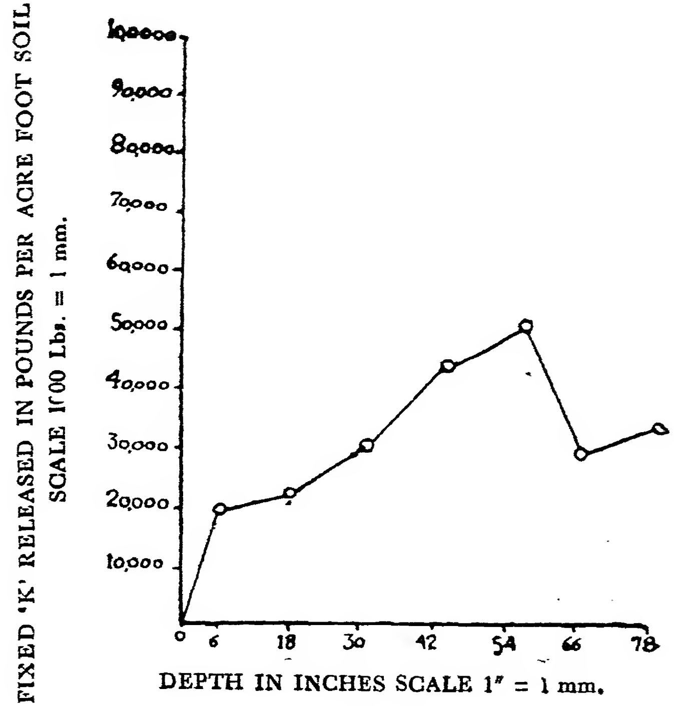

FIG. 2

RELEASE OF FIXED 'K' IN ALKALI SOIL PROFILE BY N NH_4 Cl

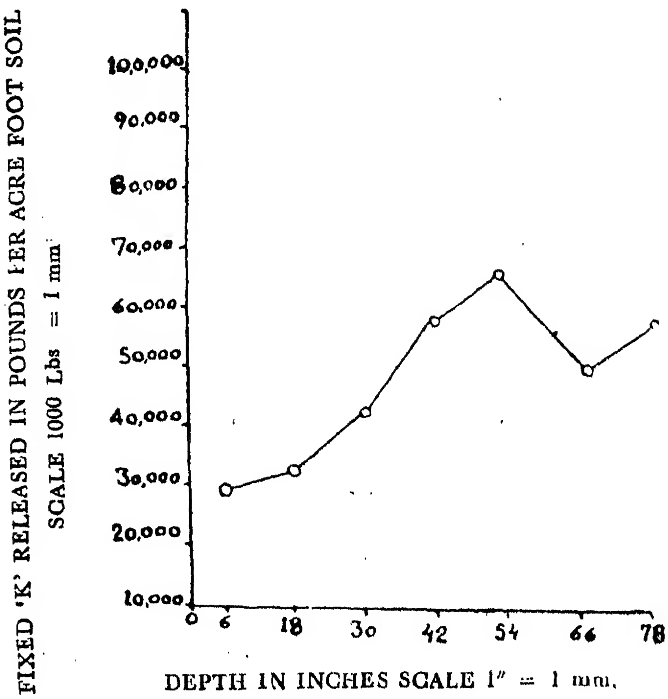

FIG. 3

DISCUSSION

The foregoing results recorded in table 2 show that the fixed K is present in the profile in all the horizons, which can easily be released with neutral N NH_4–Ac and NH_4Cl solutions, and not by N NaCl or 0·1N HCl or even by N HCl. Hoagland and Martin[10] and Joffe and Levine[15] could not obtain any release of fixed potassium by prolonged treatment with hot 10 N HCl. It indicates that H and Na ions are not able to enter the interior of the crystal lattice of the colloidal matter or they are not able to pass into the lattice " hole " as pointed out by Page and Baver[5], while NH_4 ions can pass through the lattice "hole" and replace the fixed K ions. The release of amount of Ca and Mg by 0·1N HCl exceeds the amount released with N NH_4–AC and N NH_4Cl and N NaCl. Release of Ca and Mg by N HCl is much greater. There is no release of sesquioxide by N NH_4–AC, N NH_4 Cl, 0·1N HCl and N NaCl, but, there is considerable release of R_2O_3 with N HCl. There is much release of K with normal HCl as compared with 0·1N HCl which is due to the breakdown of the colloidal complex.

Joffe and Levine[15] used normal Na_2CO_3 in releasing the fixed K ions and found no release of K and R_2O_3. This further points out that the fixation of K in the alkaline conditions in the soil is more pronounced than in the acidic conditions. Also, alkali soils are more susceptible to K fixation than the acidic soils. It is clear

from table 1 that the release of K is greater in the fifth horizon but there is a decrease in cation exchange capacity. This indicates that some of the positions in the exchange complex must have been occupied by the fixed K ions which has lowered the exchange capacity. Levine and Joffe[16] also showed that with increasing K fixation, the decrease in the exchange capacity becomes larger.

The results recorded in table 3 show that the amount of adsorbed K obtained by extracting the soil with N NH_4-AC is slightly greater than that obtained by extracting with NH_4Cl. Hoagland and Martin[10] and Chaminade[2] reported that the nature of anions exerts no influence on K fixation.

The values of exchangeable K recorded in table 4 are the amounts of K released in lbs per acre foot of soil by extracting the soil with N NaCl solution.

By substracting the values of exchangeable K (table 2) from the values of adsorbed K (Table 3), the values of fixed K were calculated in lbs per acre foot of soil and are recorded in table 6. The values of fixed K released in lbs per acre foot of soil were plotted against the depth in inches and are shown in Fig. 2 and Fig. 3 for NH_4-AC and NH_4Cl respectively. There is a great similarity between Fig. 2 and Fig. 3 indicating that the amount of K released by NH_4-AC and NH_4Cl does not differ much. It can be observed from table 4. and table 6 as also from Fig. 2, Fig. 3 and Fig. 1 that there is a great variation in the amount of fixed and exchangeable K between the horizons. This is due to the type of minerals present in them.

From table 7, it is clear that there is a considerable release of the K which was not released by extracting with ·01N HCl and N NaCl, and the amount of K released is just equivalent to the amount released by directly leaching with neutral N NH_4Cl when the amount of K released by 0·1N HCl and N NaCl and the amount of K released subsequently by leaching with N NH_4Cl are added together.

Results recorded in table 8 represent the gross and net negative charges and the negative charge on the exchange complex, *i.e.* the exchange capacity. Gross negative charge is the amount of NH_4^+ released by extracting the NH_4Cl saturated soil with N KCl and the negative charge at the exchange complex is the amount of NH_4^+ replaced by leaching with N NaCl. The net negative charge is the difference between the gross negative charge and the exchange complex. The value of net negative charge is just equivalent to the amount of K released with a small variation. This shows that fixed K and NH_4 can be replaced by each other and not by any other ions.

REFERENCES

1. Volk, N. J. *Soil Sci.*, 37 : 267–287, (1934).
2. Chaminade, R. *Compt. rend.*, 203 : 682–684, (1936).
3. Truog, E. and Jones, R. J. *Ind Eng Chem.*, 30 : 882-885, (1938).
4. Kolodny, L. *Ph.D. Thesis, Rutgers Univ , New Brunswick, New Jersey*, (1933).
5. Page, J. B. and Baver, L. D *Soil Sci. Soc. Amer. Proc.* (1939); 4 : 150–155, (1940).
6. Stanford G. and Pierre, W. H *Soil Sci. Amer. Proc.*, (1946); 11 : 155–160, (1947)
7. Stanford, G. *Soil Sci. Amer. Proc.*, (1947): 12 : 167-171, (1948).
8. Sears, O. H. *Soil Sci*, 30 : 325-347, (1930).
9. Allway, H. and Pierre, W. H. *J. Amer. Soc. Agron.*, 31 : 940–953, (1939).
10. Hoagland, D. R. and Martin, J. C. *Soil Sci. Trans., 3rd Internat. Congr. Soil Sci.*, 1 : 99–103, (1935).
11. Blume, J. M. and Purvis, E. R. *J. Amer. Soc. Agron.*, 31 : 857–863, (1939).
12. Breazeale, J. F. and Magistad, D. C. *Univ. Ariz. Tech. Bull*, 24 : (1928).
13. Piper, C. S. *Soil* and *Plant Analysis Univ. of Adelaide*, 83–85, (1947).
14. Wright, C. H. *Soil Analysis,' Thom Murby & Co., London*, (1934)
15. Joffe, J. S. and Levine, A. K. *Soil Sci.*, 62 : 411–420, (1946).
16. Levine, A. K. and Joffe, J. S. *Soil Sci.*, 63 : 329–335, (1947).

ON DUAL INTEGRAL EQUATIONS

By

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Let us consider the dual integral equations

(1) $$\int_0^\infty G(\alpha)\,\phi(\alpha)\,(\alpha\, x)^{\nu+1} J_\nu(\alpha\, x)\, d\alpha = f(x), \qquad 0 < x < 1,$$

(2) $$\int_0^\infty \phi(\alpha)\,(\alpha\, x)^{\nu+1} J_\nu(\alpha\, x)\, d\alpha = 0, \qquad x > 1,\ \nu > -\tfrac{3}{2},$$

where $G(\alpha)$ and $f(x)$ are prescribed and $\phi(\alpha)$ is an unknown function to be determined. In this note, we shall give a different way of solving equations of this type. It will be shown that, by using a method analogous to that of Lebedev and Ufland[2,3], these integral equations can be reduced to a Fredholm integral equation of second kind which can be solved by well known methods. The analysis given here is purely formal.

We set

(3) $$\phi(\alpha) = \int_0^1 g(\xi)\,(\alpha\,\xi)^{-\nu-1/2} J_{\nu+1/2}(\alpha\,\xi)\, d\xi,$$

where $g(\xi)$ is a function to be determined. Since [1, p. 48 (7)]

$$\int_0^\infty (\alpha x)^{1/2} J_\nu(\alpha\, x)\, J_{\nu+1/2}(\alpha\,\xi)\, d\alpha = (2/\pi)^{1/2}\,(x/\xi)^{\nu+1/2}\,(\xi^2 - x^2)^{-1/2} H(\xi - x), \ \nu > -1,$$

where $H(t)$ is Heaviside's unit function, the equation (2) is automatically satisfied. On integrating (1) with respect to x from 0 to x, we get

(4) $$F(x) = \int_0^x f(x)\, dx = \int_0^\infty G(\alpha)\,\phi(\alpha)\,\frac{(x\,\alpha)^{\nu+1}}{\alpha} J_{\nu+1}(\alpha\, x)\, d\alpha.$$

Let us represent the function $G(\alpha)$ in the form

(5) $$G(\alpha) = \alpha\,[1 + R(\alpha)].$$

This can always be done in practice, it being assumed that $R(\alpha) = \dfrac{G(\alpha)}{\alpha} - 1$.

Now substituting the values of $\phi(\alpha)$ and $G(\alpha)$ from (3) and (5) in (4), we get

(6) $$\int_0^\infty \int_0^1 g(\xi)\,(x/\xi)^{\nu+1/2}\,(x\,\alpha)^{1/2} J_{\nu+1}(\alpha\, x)\, J_{\nu+1/2}(\alpha\,\xi)\, d\xi\, d\alpha$$
$$+ \int_0^\infty R(\alpha) \int_0^1 g(\xi)\,(x/\xi)^{\nu+1/2}\,(x\,\alpha)^{1/2} J_{\nu+1}(\alpha\, x)\, J_{\nu+1/2}(\alpha\,\xi)\, d\xi\, d\alpha = F(x), \ x < 1.$$

The first integral in (6) is equivalent to

$$(7)\qquad \int_0^1 g(\xi)\,(x/\xi)^{\nu+1/2}\,d\xi \int_0^\infty (\alpha\, x)^{1/2} J_{\nu+1}(\alpha\, x)\, J_{\nu+1/2}(\alpha\,\xi)\,d\alpha$$

$$= (2/\pi)^{1/2} \int_0^x \frac{g(\xi)}{(x^2-\xi^2)^{1/2}}\,d\xi.$$

Here we have used the result [1, p. 47 (8)]

$$\int_0^\infty (\alpha x)^{1/2} J_{\nu+1}(\alpha\, x)\, J_{\nu+1/2}(\alpha\,\xi)\,d\alpha = (2/\pi)^{1/2}\,(\xi/x)^{\nu+1/2}\,(x^2-\xi^2)^{-1/2}\,H(x-\xi),\ \nu-\tfrac{3}{2}$$

The second integral of (6) can be written as

$$(8)\qquad \int_0^1 g(\xi)\,(x/\xi)^{\nu+1/2}\,d\xi \int_0^\infty R(\alpha)\,(\alpha\, x)^{1/2} J_{\nu+1}(\alpha\, x)\, J_{\nu+1/2}(\alpha\,\xi)\,d\alpha.$$

Since

$$J_{\nu+1/2}(\alpha\,\xi) = (2/\pi)^{1/2} \int_0^\xi (u/\xi)^{\nu+1/2} \frac{(\alpha\, u)^{1/2} J_\nu(\alpha\, u)}{(\xi^2-u^2)^{1/2}}\,du,$$

the second integral of (8) is equivalent to

$$(2/\pi)^{1/2} \int_0^\xi \frac{(u/\xi)^{\nu+1/2}}{(\xi^2-u^2)^{1/2}} \left(\int_0^\infty R(\alpha)\,(x\,\alpha)^{1/2} J_{\nu+1}(\alpha\, x)\,(\alpha\, u)^{1/2} J_\nu(\alpha\, u)\,d\alpha \right) du$$

$$= 2/\pi \int_0^\xi \frac{(x/\xi)^{\nu+1/2}\,\bar{H}(u,x)}{(\xi^2-u^2)^{1/2}}\,du,$$

where

$$H(u,x) = (\pi/2)^{1/2}\,(u/x)^{\nu+1/2} \int_0^\infty R(\alpha)\,(x\,\alpha)^{1/2} J_{\nu+1}(\alpha\, x)\,(\alpha\, u)^{1/2} J_\nu(\alpha\, u)\,d\alpha.$$

Hence equation (8) can be replaced by

$$(9)\qquad 2/\pi \int_0^1 g(\xi)\,(x/\xi)^{2\nu+1} \left(\int_0^\xi \frac{\bar{H}(u,x)}{(\xi^2-u^2)^{1/2}}\,du \right) d\xi.$$

Substituting the values of the first and second integrals of (6) from (7) and (9), we obtain

$$(10)\qquad \int_0^x \frac{g(\xi)}{(x^2-\xi^2)^{\frac{1}{2}}}\,d\xi + 2/\pi \int_0^1 g(\xi)\,(x/\xi)^{2\nu+1} \left(\int_0^\xi \frac{\bar{H}u, x)}{(\xi^2-u^2)^{\frac{1}{2}}}\,du \right) d\xi = F(x), \qquad x<1.$$

We know that Abel's integral equation

$$(11)\qquad h(x) = \int_0^x \frac{g(\xi)}{(x^2-\xi^2)^{\frac{1}{2}}}\,d\xi$$

has the solution

$$(12)\qquad g(\xi) = 2/\pi \frac{d}{d\xi} \int_0^\xi \frac{x\, h(x)}{(\xi^2-x^2)^{\frac{1}{2}}}\,dx.$$

Using (11) and (12), in equation (10) we obtain

$$h(x) + 4/\pi^2 \int_0^1 \frac{d}{d\xi}\left(\int_0^{\xi} \frac{w\, h(w)}{(\xi^2 - w^2)^{\frac{1}{2}}} dw\right) M(x, \xi)\, d\xi = F(x), \quad x < 1, \tag{13}$$

where

$$M(x,\xi) = (x/\xi)^{2\nu+1} \int_0^{\xi} \frac{H(u, x)}{(\xi^2 - u^2)^{\frac{1}{2}}}\, du.$$

Integrating by parts, we get

$$\int_0^1 M(x, \xi) \frac{d}{d\xi}\left(\int_0^{\xi} \frac{w\, h(w)}{(\xi^2 - w^2)^{\frac{1}{2}}} dw\right) d\xi \tag{14}$$

$$= \left[M(x, \xi) \int_0^{\xi} \frac{w\, h(w)}{(\xi^2 - w^2)^{\frac{1}{2}}}\, dw \right]_0^1 - \int_0^1 M'_{\xi}(x, \xi) \int_0^{\xi} \frac{w\, h(w)}{(\xi^2 - w^2)^{\frac{1}{2}}}\, dw\, d\xi$$

$$= M(x, 1) \int_0^{\xi} \frac{w\, h(w)}{(1 - w^2)^{\frac{1}{2}}}\, dw - \int_0^1 w\, h(w)\, dw \int_w^1 \frac{M'_{\xi}(x, \xi)}{(\xi^2 - w^2)^{\frac{1}{2}}}\, d\xi$$

$$= \int_0^1 h(w)\, K(x, w)\, dw.$$

where

$$K(x, w) = w \left[\frac{M(x, 1)}{(1 - w^2)^{\frac{1}{2}}} - \int_w^1 \frac{M'_{\xi}(x, \xi)}{(\xi^2 - w^2)^{\frac{1}{2}}}\, d\xi\right]$$

This finally yields

$$h(x) + 4/\pi^2 \int^1 h(w)\, K(x, w)\, dw = F(x), \; x < 1. \tag{15}$$

This is Fredholm integral equation of the second kind with a kernel which is weakly singular, therefore the classical theory for such equations can be applied for finding $h(x)$. Knowing $h(x)$, we can calculate $g(\xi)$ from (12) and hence $\phi(\alpha)$ can be calculated from (3).

REFERENCES

1. Erdelyi, A. *et al.* Tables of integral transforms. *McGraw-Hill, New York*, 2 : (1954).
2. Lebedev, N. N. *Dokl. Akad, Nauk SSSR*, 114, (1957).
3. Lebedev, N. N. and Ufland, Y. S. *Prikl. Mat. Meh.* 22, (1958).

ON THE STRUCTURE OF THE BARRED GALAXIES

By

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ABSTRACT

The possibility of the motion of gas, for a barred galaxy, under the influence of gravitational force, has been investigated. A homogeneous cylinder, rotating uniformly, has been assumed to represent a barred galaxy. Another rotation model, a homogeneous elliptic cylinder has also been considered for the barred galaxy. We have dealt with the problem assuming the resistance, due to the medium, encountered by the gas, to vary as the velocity. For both the models the cartesian coordinates, x and y have been determined as a function of the parameter 't'.

Numerical solutions, for the first model, confirm the outward flow of gas. The case has been treated when motion commences from a point on the axis of the bar. The time, for the outflow of gas, from $x_0 = 0$, is found to be $\sim 1{\cdot}5 \times 10^8$ yrs.

INTRODUCTION

Burbidge *et al*,[1-4] have considered a prolate spheroid of uniform density, in uniform rotation, to represent a barred galaxy. They found the model to be convenient in determining the mass of the barred galaxy. The model appears to be an oversimplification, considering the central condensation and the density distribution of matter in the bar. Aarseth[5,6] has considered a sphere with symmetrical cylindrical bars, all being of same density and in uniform rotation, to represent a barred galaxy. He has determined the density distribution in the bar and has obtained some homology relations. Ôki *et al*[7] have discussed the outward flow of gas, along the bar, and the formation of trailing and leading spiral arms, for the Aarseth's model. They have suggested that the θ-shaped barred galaxies may be transient features appearing at a late stage of the evolution sequence. As an improvement, in connection with the dark lanes of the galaxies, they have proposed a composite model of the galaxy. It consists of the prolate spheroid main body and the oblate spheroid disc.

In the present paper a somewhat simple model has been considered. We have proposed a cylinder of uniform density rotating in a solid wheel like manner, for the barred galaxy. This model has been chosen, because the Large Magellanic Cloud, (previously considered to be an irregular galaxy, but now generally supposed to possess a barred symmetry) can well be considered as a rotating cylinder the axis of which is perpendicular to the direction of rotation. An analytical solution, to the motion of gas element, was not possible for the sphere-cylinder model of Aarseth. The orbit was deduced under certain assumptions. It can be seen, however, for our model exact numerical solutions are possible.

Basic Equations :

Case I.—Cylindrical Model.

The equations, in rotating coordinates, under gravitational forces alone, are given by[8]

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$$\left.\begin{aligned} \frac{d^2x}{dt^2} - 2\Omega\frac{dy}{dt} - \Omega^2 x &= X_c \\ \frac{d^2y}{dt^2} + 2\Omega\frac{dx}{dt} - \Omega^2 y &= Y_c \end{aligned}\right\} \quad (1)$$

where Ω is the angular velocity of uniform rotation, of the barred galaxy, about the z-axis. The x-y axes are the two perpendicular axes to it, which lie in the equatorial plane.

The expression for the x-component of the gravitational force, due to the cylindrical bar, following Ôki *et al*[7], is

$$X_c = -2\pi G \rho x \quad (2)$$

G being the gravitational constant and ρ the mean density of the bar.

The following condition is always required on the y-axis of the bar

$$Y_c + \Omega^2 y = 0 \quad (3)$$

Hence from equations (1) and (3)

$$\frac{d^2y}{dt^2} = -2\Omega\frac{dx}{dt} \quad (4)$$

Integrating equation (4) and applying initial conditions,

when $x = x_0$, $\frac{dy}{dt} = V_c$, we have

$$\frac{dy}{dt} = V_c - 2\Omega(x - x_0) \quad (5)$$

From equations (1), (2) and (5)

$$\frac{d^2x}{dt^2} = -(2\pi G\rho + 3\Omega^2)x + (2\Omega V_c + 4\Omega^2 x_0) \quad (6)$$

The solution of the above equation is

$$x = C_1 \cos(2\pi G\rho + 3\Omega^2)^{\frac{1}{2}} t + C_2 \sin(2\pi G\rho + 3\Omega^2)^{\frac{1}{2}} t + \frac{2\Omega V_c + 4\Omega^2 x_0}{2\pi G\rho + 3\Omega^2} \quad (7)$$

where C_1 and C_2 are arbitrary constants given by

$$\left.\begin{aligned} C_1 &= \frac{2\pi G\rho x_0 - 2\Omega V_c - \Omega^2 x_0}{2\pi G\rho + 3\Omega^2} \\ C_2 &= 0 \end{aligned}\right\} \quad (8)$$

The complete solution of equation (6) is

$$x = \left(\frac{2\pi G\rho x_0 - 2\Omega V_c - \Omega^2 x_0}{2\pi G\rho + 3\Omega^2}\right)\cos(2\pi G\rho + 3\Omega^2)^{\frac{1}{2}} t$$

$$+ \frac{2\Omega V_c + 4\Omega^2 X_0}{2\pi G\rho + 3\Omega^2} \tag{9}$$

From equations (5) and (9) we find on simplification

$$\frac{dy}{dt} = \frac{(V_c + 2\Omega x_0)(2\pi G\rho - \Omega^2)}{(2\pi G\rho + 3\Omega^2)}$$

$$+ \frac{2\Omega(\Omega^2 x_0 + 2\Omega V_c - 2\pi G\rho x_0)}{2\pi G\rho + 3\Omega^2}\cos(2\pi G\rho + 3\Omega^2)^{\frac{1}{2}} t \tag{10}$$

Let us assume

$$\left.\begin{aligned} a &= \frac{(V_c + 2\Omega x_0)(2\pi G\rho - \Omega^2)}{2\pi G\rho + 3\Omega^2} \\ b &= \frac{2\Omega(\Omega^2 x_0 + 2\Omega V_c - 2\pi G\rho x_0)}{2\pi G\rho + 3\Omega^2} \end{aligned}\right\} \tag{11}$$

and $$c = (2\pi G\rho + 3\Omega^2)^{\frac{1}{2}}$$

Thus on integrating we have

$$y = D + at + \frac{b}{c}\sin ct. \tag{12}$$

Now when $t = 0, y = R_1$. Hence $D = R_1$ and equation (12) is thus

$$y = R_1 + \frac{(V_c + 2\Omega x_0)(2\pi G\rho - \Omega^2)}{2\pi G\rho + 3\Omega^2}\, t$$

$$+ \frac{2\Omega(\Omega^2 x_0 + 2\Omega V_c - 2\pi G\rho x_0)}{(2\pi G\rho + 3\Omega^2)^{3/2}}\sin(2\pi G\rho + 3\Omega^2)^{\frac{1}{2}} t \tag{13}$$

The orbit of the gas element can easily be determined with the help of parametricequations (9) and (13).

Case II.—Cylindrical Model with resistance

The gas element, in its outward motion, is bound to experience some resistance due to the presence of pervading gas and dust. In the present analysis the resistance, due to the medium, is taken[9] as Lv where L is a small constant. The basic equations, in rotating coordinates, under gravitational and resisting force, are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} - 2\Omega\frac{dy}{dt} - \Omega^2 x &= X_c - L\frac{dx}{dt} \\ \frac{d^2y}{dt^2} + 2\Omega\frac{dx}{dt} - \Omega^2 y &= Y_c - L\frac{dy}{dt} \end{aligned}\right\} \tag{14}$$

where as before $X_c = -2\pi G\rho x$, the other symbols have their usual meaning.

The following condition is always required on the y-axis of the bar

$$Y_c + \Omega^2 y - L\frac{dy}{dt} = 0 \tag{15}$$

Hence $\qquad \dfrac{d^2y}{dt^2} = -2\,\Omega\,\dfrac{dx}{dt}$

On integrating above equation, and applying the conditions, we find

$$\frac{dy}{dt} = V_c - 2\,\Omega\,(x - x_0) \tag{16}$$

From equations (14) and (16) we have

$$\frac{d^2x}{dt^2} + L\,\frac{dx}{dt} + (2\,\pi\,G\,\rho + 3\Omega^2)\,x = (2\Omega V_c + 4\,\Omega^2\,x_0) \tag{17}$$

The solution of equation (17) is easily seen to be given by

$$x = e^{-\frac{L}{2}t}\left\{D_1 \cos \frac{\sqrt{4\,(2\,\pi\,G\,\rho + 3\,\Omega^2) - L^2}}{2}\,t + D_2 \sin \frac{\sqrt{4(2\pi G\,\rho + 3\Omega^2) - L^2}}{2}\,t\right\}$$
$$+ \frac{2\Omega V_c + 4\Omega^2\,x_0}{2\,\pi\,G\,\rho + 3\Omega^2} \tag{18}$$

where D_1 and D_2 are arbitrary constants, the value of which is given by

$$\left.\begin{aligned} D_1 &= \frac{2\,\pi\,G\,\rho\,x_0 - 2\,\Omega\,V_c - \Omega^2\,x_0}{2\,\pi\,G\,\rho + 3\Omega^2} \\ D_2 &= \frac{L\,(2\,\pi\,G\,\rho\,x_0 - 2\,\Omega V_c - \Omega^2\,x_0)}{(2\,\pi\,G\,\rho + 3\Omega^2)\,(\sqrt{4\,(2\,\pi\,G\,\rho + 3\Omega^2) - L^2})} \end{aligned}\right\} \tag{19}$$

Hence the complete solution of equation (17) is

$$x = e^{-\frac{L}{2}t}\left\{\left(\frac{2\,\pi\,G\,\rho\,x_0 - 2\,\Omega V_c - \Omega^2\,x_0}{2\,\pi\,G\,\rho + 3\Omega^2}\right) \cos \frac{\sqrt{4\,(2\,\pi\,G\,\rho + 3\Omega^2) - L^2}}{2}\,t\right\}$$
$$+ e^{-\frac{L}{2}t}\left\{\frac{L(2\,\pi\,G\,\rho\,x_0 - 2\,\Omega\,V_c - \Omega^2\,x_0)}{(2\,\pi\,G\,\rho + 3\,\Omega^2)\,(\sqrt{4\,(2\pi\,G\,\rho + 3\,\Omega^2) - L^2}} \quad \sin \frac{\sqrt{4(2\pi G\rho + 3\Omega^2) - L^2}}{2}\,t\right\}$$
$$+ \frac{2\,\Omega\,V_c + 4\,\Omega^2\,x_0}{2\,\pi\,G\,\rho + 3\,\Omega^2} \tag{20}$$

From equations (16) and (20) we finally have

$$\frac{dy}{dt} = \frac{(V_c + 2\Omega\,x_0)\,(2\,\pi\,G\,\rho - \Omega^2)}{2\,\pi\,G\,\rho + 3\,\Omega^2}$$

$$+ \left\{\frac{2\Omega(\Omega^2\,x_0 + 2\,\Omega V_c - 2\,\pi\,G\,\rho\,x_0)}{(2\,\pi\,G\,\rho + 3\,\Omega^2)}\,e^{-\frac{L}{2}t} \cos \frac{\sqrt{4\,(2\,\pi\,G\,\rho + 3\,\Omega^2) - L^2}}{2}\right.,$$

$$+ \left\{ \frac{2\,\Omega\, L\,(\Omega^2 x_0 + 2\,\Omega\, V_c - 2\pi\, G\rho\, x_0)}{(\sqrt{4\,(2\pi\, G\rho + 3\Omega^2) - L^2})\,(2\pi\, G\rho + 3\,\Omega^2)} \right\} e^{-\frac{L}{2}t} \sin \frac{\sqrt{4\,(2\pi\, G\rho + 3\Omega^2) - L^2}}{2}\,t \tag{21}$$

$$\left.\begin{aligned}
\text{Let}\quad a_1 &= \frac{(V_c + 2\,\Omega x_0)\,(2\pi\, G\rho - \Omega^2)}{2\pi\, G\rho + 3\Omega^2} \\
b_1 &= \frac{2\Omega\,(\Omega^2 x_0 + 2\Omega V_c - \pi\, G\rho\, x_0)}{2\pi\, G\rho + 3\Omega^2} \\
b'_1 &= \frac{2\Omega\, L\,(\Omega^2 x_0 + 2\Omega V_c - 2\pi G\, \rho x_0)}{(2\pi\, G\rho + 3\Omega^2)\,(\sqrt{4\,(2\pi G\rho + 3\Omega^2) - L^2})} \\
\text{and}\quad c_1 &= \frac{\sqrt{4\,(2\pi G\rho + 3\Omega^2) - L^2}}{2}
\end{aligned}\right\} \tag{22}$$

$$\text{Hence } y = \int \left\{ a_1 + b_1\, e^{-\frac{L}{2}t} \cos c_1 t + b_1'\, e^{-\frac{L}{2}t} \sin c_1 t \right\} dt \tag{23}$$

$$y = F + a_1 t + \left(\frac{b_1 c_1 - \frac{L}{2} b_1'}{\frac{L^2}{4} + c_1^2} \right) e^{-\frac{L}{2}t} \sin c_1 t - \left(\frac{b_1 \frac{L}{2} + b_1' c_1}{\frac{L^2}{4} + c_1^2} \right) e^{-\frac{L}{2}t} \cos c_1 t \tag{24}$$

$$\text{When } t = 0,\ y = R_1 \text{ hence } F = R_1 + \frac{b_1 \frac{L}{2} + b_1' c_1}{\frac{L^2}{4} + c_1^2} \tag{25}$$

$$\begin{aligned}
y = R_1 + \left(\frac{b_1 \frac{L}{2} + b'_1 c_1}{\frac{L^2}{4} + c^2_1} \right) + a_1 t + \left(\frac{b_1 c_1 - \frac{L}{2} b_1'}{\frac{L^2}{4} + c_1^2} \right) e^{-\frac{L}{2}t} \sin c_1 t \\
- \left(\frac{b_1 \frac{L}{2} + b_1' c_1}{\frac{L^2}{4} + c_1^2} \right) e^{-\frac{L}{2}t} \cos c_1 t
\end{aligned} \tag{26}$$

where the values of a_1, b_1, b_1' and c_1 are given by equation (22).

It is interesting to see that if $L = 0$ then we get the equation determined before, when resistance is not considered, namely

$$y = R_1 + at + (b/c) \sin ct \tag{27}$$

The orbit of the gas element, when resistance is considered, can be deduced with the help of equations (20) and (26).

Elliptic Cylinder Model.

It may be observed that if we choose a homogeneous elliptic cylinder, to represent a barred galaxy, then the expressions for x and y in terms of the parameter 't' will be very much the same. The alterations will be in the value of the constants occurring therein. The value of the constants a, b and c in equation (11) will be given by

$$\left.\begin{aligned}
a &= \frac{(V_c + 2\,\Omega x_0)\left(\dfrac{4\pi\, G\rho\,\beta}{\alpha+\beta} - \Omega^2\right)}{\dfrac{4\pi\, G\rho\beta}{\alpha+\beta} + 3\,\Omega^2} \\
b &= \frac{2\,\Omega\left(\Omega^2\, x_0 + 2\,\Omega\, V_c - \dfrac{4\pi\, G\rho\,\beta}{\alpha+\beta}\, x_0\right)}{\dfrac{4\pi\, G\rho\,\beta}{\alpha+\beta} + 3\,\Omega^2} \\
c &= \left(\frac{4\,\pi\, G\,\rho\,\beta}{\alpha+\beta} + 3\,\Omega^2\right)^{\frac{1}{2}}
\end{aligned}\right\} \tag{28}$$

and the value of the constants a_1, b_1, b'_1 and c_1 occurring in equation (22) as

$$\begin{aligned}
a_1 &= \frac{(V_c + 2\,\Omega\, x_0)\left(\dfrac{4\pi\, G\rho\,\beta}{\alpha+\beta} - \Omega^2\right)}{\dfrac{4\pi\, G\rho\,\beta}{\alpha+\beta} + 3\,\Omega^2} \\
b_1 &= \frac{2\,\Omega\left(\Omega^2\, x_0 + 2\,\Omega\, V_c - \dfrac{4\pi\, G\rho\,\beta\, x_0}{\alpha+\beta}\right)}{\dfrac{4\pi\, G\rho\,\beta}{\alpha+\beta} + 3\,\Omega^2} \\
b_1' &= \frac{2\,\Omega\, L\left(\Omega^2\, x_0 + 2\,\Omega\, V_c - \dfrac{4\pi\, G\rho\,\beta\, x_0}{\alpha+\beta}\, x_0\right)}{\left(\dfrac{4\pi\, G\rho\,\beta}{\alpha+\beta} + 3\,\Omega^2\right)\left[4\left(\dfrac{4\pi\, G\rho\,\beta}{\alpha+\beta} + 3\,\Omega^2\right) - L^2\right]^{\frac{1}{2}}} \\
c_1 &= \tfrac{1}{2}\left[4\left(\frac{4\pi\, G\rho\,\beta}{\alpha+\beta} + 3\,\Omega^2\right) - L^2\right]^{\frac{1}{2}}
\end{aligned} \tag{29}$$

The semi-axes of the elliptical cross section being α and β.

Numerical results.

If motion commences from a point on the axis of the cylinder, equations (9) and (13) can be written as

$$x = \frac{2\,\Omega\, V_c}{2\,\pi\, G\,\rho + 3\,\Omega^2}\left[1 - \cos\,(2\pi\, G\rho + 3\,\Omega^2)^{\frac{1}{2}}\, t\right] \tag{30}$$

and

$$y = R_1 + \frac{V_0(2\,\pi\, G\,\rho - \Omega^2)}{2\,\pi\, G\,\rho + 3\,\Omega^2}\, t + \frac{4\,\Omega^2\, V_c}{(2\,\pi\, G\,\rho + 3\,\Omega^2)^{3/2}}\, \sin\,(2\pi\, G\,\rho + 3\,\Omega^2)^{\frac{1}{2}}\, t \tag{31}$$

Assuming the following model[10]

$$\left.\begin{aligned}
&\Omega = 1{\cdot}43 \times 10^{-15}\ \text{radian/sec, Width of the bar}=2\ \text{kpc} \\
&R_1 = 3\ \text{kpc},\ R_2 = 8\ \text{kpc, Mean density of the bar, } \rho = 4{\cdot}2\times 10^{-23}\ gm/cm^3
\end{aligned}\right\} \tag{32}$$

We find

$$\left.\begin{aligned} \frac{2\,\Omega\,V_c}{2\,\pi\,G\,\rho+3\,\Omega^2} &= \cdot 1957\,;\ (2\,\pi\,G\,\rho+3\,\Omega^2)^{1/2} = 1\cdot 538\times 10^{-7} \\ \frac{V_c(2\pi\,G\,\rho-\Omega^2)}{2\,\pi\,G\,\rho+3\,\Omega^2} &= 3\cdot 361\times 10^{-8};\quad \frac{4\,\Omega^2\,V_c}{(2\,\pi\,G\,\rho+3\,\Omega^2)^{3/2}} = \cdot 1149 \end{aligned}\right\} \tag{33}$$

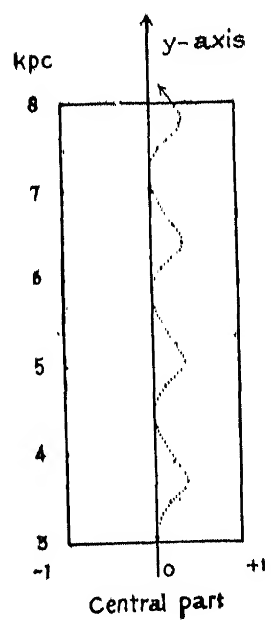
Table I gives the value of x and y for different values of t

TABLE I

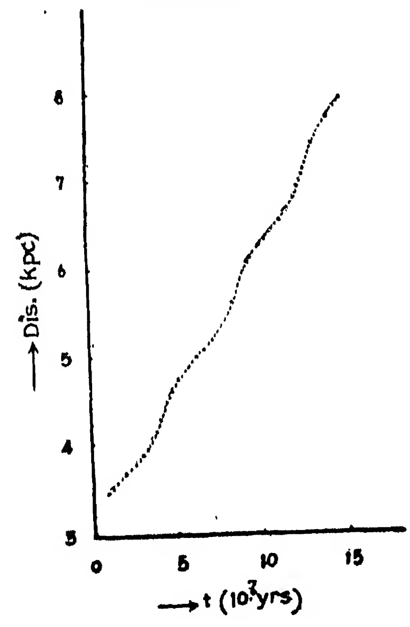
t (yrs)	x (kpc)	y (kpc)	t (yrs)	x (kpc)	y (kpc)
10^6	·00	3·05	8×10^7	·01	5·66
1×10^7	·19	3·45	9×10^7	·14	6.13
2×10^7	·39	3·68	10×10^7	·38	6·40
3×10^7	·22	3·89	11×10^7	·27	6·59
4×10^7	·00	4·33	12×10^7	·02	6·99
5×10^7	·16	4·79	13×10^7	·11	7·47
6×10^7	·39	5·04	14×10^7	·37	7·76
7×10^7	·24	5·24	15×10^7	·29	7·94

Graph I shows the outward motion of gas with increasing time, Graph II shows the variation of distance, from the point of commencement of motion, of the gas element, against time.

GRAPH I



GRAPH II



The above calculations were made by hand and are thus subject to numerical error, but it is not thought likely that they will be of significance since every precaution has been taken to avoid them.

It can be argued that for $V_0 > 0$ the gas will generally travel outwards along the bar and may even constitute the leading or trailing spiral arms. The dark lanes of the barred galaxies, which represent the inward motion of gas towards the nucleus, can easily be accounted if $V_0 < 0$. The inward motion of gas may be due to the exchange of angular momentum between the bar and the gas streaming outwards.

In another communication it is proposed to discuss motion of gas for other values of x_0. The effect of resistance will also be considered, by taking different resistance laws. The spheroidal-cylinder model for the barred galaxy will also be discussed, since a slightly flattened central nucleus will be a more realistic choice, for the rotating system.

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REFERENCES

1. Burbidge, E. M. and Burbidge, G. R. *Ap. J.* 132 : (1960).
2. Burbidge, E M. *et al.*, *ibid*, 132 : 654–660, (1960).
3. Burbidge, E. M *et al.* *ibid*, 132 : 661–666, (1960).
4. Burbidge, E. M. *et al.*, *ibid*, 136 : (1962).
5. Aarseth, S. J. *Month. Not. Roy. Astron. Soc*, 121 : 525–529, (1960)
6. Aarseth, S. J *ibid*, 122 : 535–541, (1961).
7. Ôki, T. *et al* *Supp. Prog. Theo. Phy.*, No. 31, 77–115, (1965).
8. Routh, E. J. *A Treatise on Dynamics of a Particle. Camb. Univ. Press.* 154, (1898).
9. Lal, B. B. *Proc. Nat. Acad. Sci.*, 13(3) : 165–170, (1943).
10. Ôki, T. (Private communication).

SPECTROPHOTOMETRIC DETERMINATION OF THE STABILITY CONSTANT OF COBALT-NITROSO R-SALT CHELATE

By

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ABSTRACT

The complex formation between Co(II) and Nitroso-R-Salt (NRS) has been studied adopting the spectrophotometric method. It has been shown that a stable yellowish-red coloured chelate at pH 6·5 with λ_{max} at 400 mμ is formed. Job's method of continuous variations shows the formation of a chelate of the type 1 : 3. A new method for the determination of the stability constant has been described in detail. The value of the stability constant log K = 13·28 and free energy change ΔG^0 = – 18·23 Kcals. at 25°C have been computed for the complex.

INTRODUCTION

Nitroso R-salt (1-nitroso-2-napthol-3 : 6 disulphonate) has been extensively used as a colorimetric and photometric reagent for metals including cobalt. Spectrophotometric methods have been used in many ways to study complexes in solution and the continuous variation method[1] seems to be suitable. An extension of the system in which two or more complexes are formed has been suggested by Vosburgh and Cooper[2] but a more general treatment has been given by Katzin and Gebert[4].

This paper reports the theoretical and experimental procedure adopted for the study of the Co-NRS system which forms a 1 : 3 chelate. The determination of the stability constant of the chelate employing the continuous variation data is discussed in details. In a recent preliminary note the outline of the method adopted has been described.[4]

THEORETICAL

Anderson and coworkers[5] have described a method for the determination of the stability constant which is based on the determination of composition of solutions having an identical intensity of colour *i.e.* the same absorbance values. In this method, both the reactants should be colourless and this places a restriction on its use. Dey and coworkers[6] have modified the method to suit systems where one of the reactants may be coloured. These workers have put forward a treatment for 1 : 1 and 1 : 2 chelates but have not extended their method for 1 : 3 chelates.

An extension of the method is for the chelates where the combining ratio of the metal ligand and ion is 3 : 1. The theoretical considerations underlying the method are described as follows :

Let us consider the formation of a complex between the metal M and the chelating agent Ke, where for a 1 : 3 complex, the reaction may be represented as

$$M + 3Ke = M(Ke)_3$$

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Let the inital concentrations of M and Ke be a and b respectively and x be the equilibrium concentration of the chelate. At equilibrium, the concentrations of M and Ke will be a-x and b-$3x$ respectively. Hence the formation constant or stability constant will be given by the expression :

$$K = \frac{x}{(a-x)\,(b-3x)^3} \tag{1}$$

Now applying the expression for two concentrations *i.e.* a_1 and a_2 of M and b_1 and b_2 of Ke, at the same absorbance value of the chelate *i.e.* for the same value of x, we have

$$K = \frac{x}{(a_1-x)\,(b_1-3x)^3} = \frac{x}{(a_2-x)\,(b_2-3x)^3} \tag{2}$$

To determine the value of K, it is necessary to determine the value of x from the equation (2) in terms of a_1, b_1, a_2 and b_2. Considering only the two right hand side expressions and rearranging we have :

$$27\,[(a_1-a_2)+(b_1-b_2)]\,x^3 + 9\,[3(a_2\,b_2-a_1\,b_1)+(b_2^2-b_1^2)]\,x^2$$
$$+\,[(b_1^3-b_2^3)+9(a_1\,b_1^2-a_2 b_2^2)]\,x + [a_2\,b_2^3-a_1\,b_1^3] = 0$$

A slight rearrangement of equation (2) leads to :

$$x^3 + \frac{27(a_2\,b_2-a_1b_1)+9(b_2^2-b_1^2)}{27(a_1-a_2+b_1-b_2)}\,x^2 + \frac{(b_1^3-b_2^3)+9(a_1\,b_1^2-a_2\,b_2^2)}{27(a_1-a_2+b_1-b_2)}\,x$$
$$+\frac{a_2\,b_2^3-a_1\,b_1^3}{27(a_1-a_2+b_1-b_2)} = 0 \tag{3}$$

which may be written in a simpler manner as :

$$x^3 + A_1x^2 + A_2x + A_3 = 0 \tag{4}$$

where the coefficients A_1, A_2 and A_3 are defined by equation (3).

This equation cannot be solved as such, and is therefore reduced to a standard cubic equation :

$$Y^3 + 3\,HY + G = 0 \tag{5}$$

where $Y = x - H$, $3H = \dfrac{3A_2 - A_1^2}{3}$ and $G = A_3 + \dfrac{2A_1^3}{27} - \dfrac{A_1\,A_2}{3}$

Employing Cordon's method of solution of standard cubics, the roots of the above equation may be found to be :

$$Y_1 = \left(\frac{-G+\sqrt{G^2+4H^3}}{2}\right)^{1/3} + \left(\frac{-G-\sqrt{G^2+4H^3}}{2}\right)^{1/3}$$

$$Y_2 = \left(\frac{-G+\sqrt{G^2+4H^3}}{2}\right)^{1/3} w + \left(\frac{-G-\sqrt{G^2+4H^3}}{2}\right)^{1/3} w^2$$

and

$$Y_3 = \left(\frac{-G + \sqrt{G^2 + 4H^3}}{2}\right)^{1/3} w^2 + \left(\frac{-G - \sqrt{G^2 + 4H^3}}{2}\right)^{1/3} w$$

where w and w^2 are the complex cube roots of unity and their values are $-\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$. Hence the roots of the original equation will be

$$x_1 = Y_1 - A_{1/3}$$

$$x_2 = Y_2 - A_{1/3}$$

and

$$x_3 = Y_3 - A_{1/3}$$

Nature of the different values of x :

From the above values of x, it can be easily seen that the nature of the roots of x will depend upon numerical values of $G^2 + 4H^3$. The following cases may be mentioned.

(*i*) if $G^2 + 4H^3 > 0$, one root will be real and the other two are complex.

(*ii*) if $G^2 + 4H^3 = 0$, all the roots will be real $x_2 = x_3$

(*iii*) if $G^2 + 4H^3 < 0$, all roots will be real and different.

The values of $G^2 + 4H^3$ are computed from the different known values of a_1, a_2, b_1 and b_2. It has been found in all the cases that values of $G^2 + 4H^3$, from experimentally measured values of a_1, b_1 and a_2, b_2 are greater than zero. This shows that there will be only one real value of x in the present case and that is when,

$$x_1 = Y_1 - A_{1/3}$$

which gives :

$$x = \left(\frac{-G + \sqrt{G^2 + 4H^3}}{2}\right)^{1/3} + \left(\frac{-G - \sqrt{G^2 + 4H^3}}{2}\right)^{1/3} - A_{1/3}$$

here G, H and A_1 have already been defined.

From equation (6) the value of x is found and substituting in equation (1), the value of K is obtained.

EXPERIMENTAL

Reagents : An A. R., B. D. H. grade cobalt sulphate was used and estimated by the usual methods. Nitroso–R–salt was prepared in CO_2 free conductivity water, and was used throughout the course of the study. A standard solution of 0·1 M $NaClO_4$ (B. D. H.) was used to maintain the ionic strength of the solutions at constant level.

Apparatus : Spectrophotometric measurements were made by means of a Unicam SP 500 spectrophotometer with glass cells of 1 cm width. Hydrogen ion concentra-

tions of the solutions were measured with a Leeds and Northrup direct reading pH indicator. The experiments were carried out in an air conditioned room maintained at 25°C.

Procedure : In order to determine the number of complexes formed by the interaction of cobalt sulphate and NRS, the reactants were mixed in different proportions and the absorbance was measured at different wavelengths ranging from 360 mμ to 600 mμ. All the mixtures gave a maximum at 400 mμ showing thereby the formation of only one complex (Figure 1).

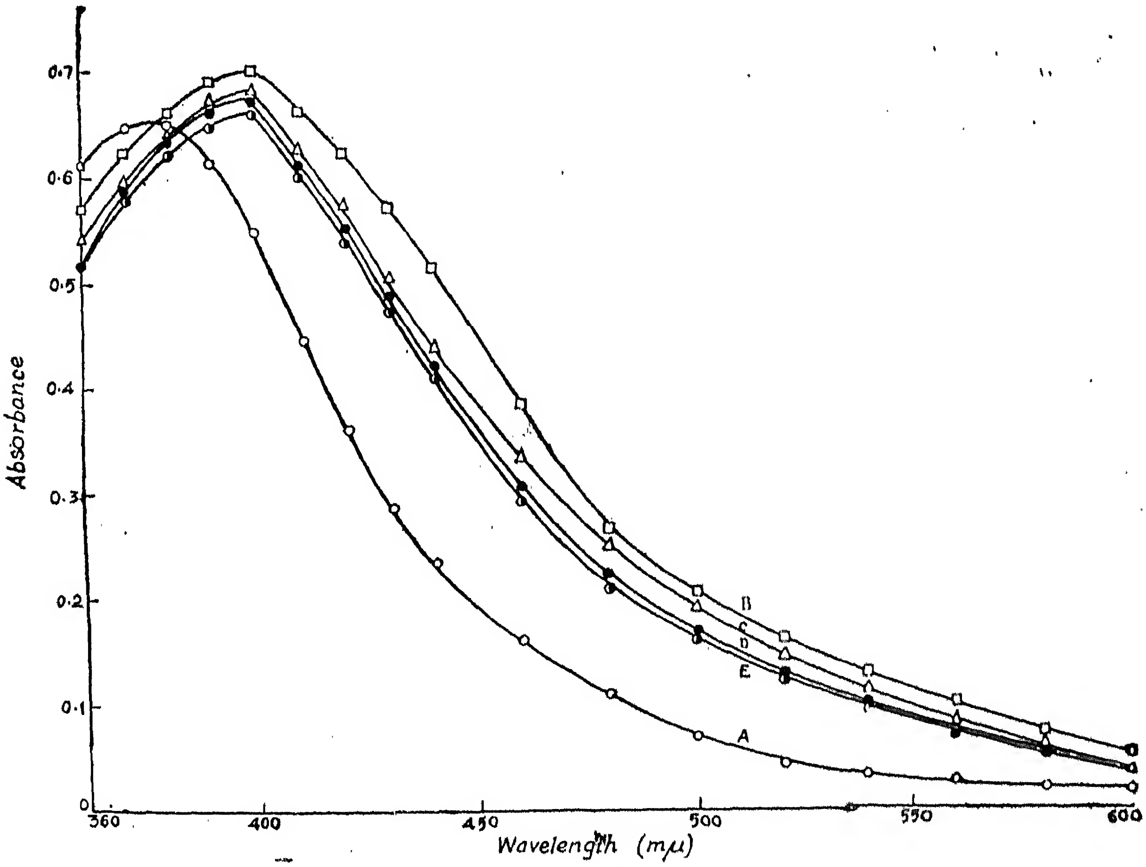

Fig. 1. Nature of the complex formed using absorbance measurements.

Final concentrations of the reactants M $\times$ 10^{-5}.

	$CoSO_4$	NRS	Ratio
A	0·00	8·0	0 : 1·0
B	16 00	8·0	1 : 0·5
C	8·00	8·0	1 : 1·0
D	4·00	8·0	1 : 2·0
E	2·66	8·0	1 : 3·0

For the determination of the composition of the chelate solutions of cobalt sulphate and nitroso R-salt of three different equimolecular concentrations *viz.* $2{\cdot}00 \times 10^{-4}$M, $1{\cdot}33 \times 10^{-4}$M and $1{\cdot}00 \times 10^{-4}$M were mixed according to the method of continuous variations and absorbance was measured at 420 mμ after fixing the pH of different mixtures at 6·5 in a total volume of 20 ml. and ionic strength at 0·1. The absorbance of NRS was also determined at 420 mμ and difference between the absorbance of the mixtures and that of NRS was plotted against the ratio Co^{++}/Co^{++}+NRS (figure 2).

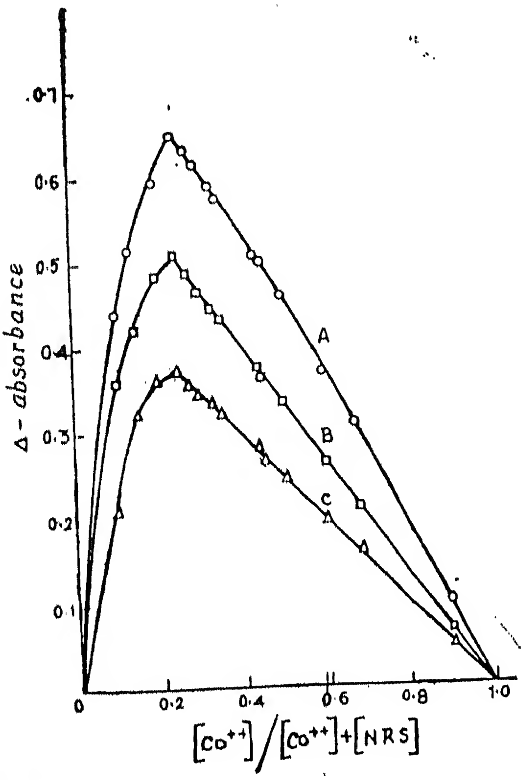

Fig. 2. Composition of the Chelate by the Continuous Variations method at pH 6·5 : where p = c'/c = 1 (c'=Conc. of the ligand and c=Conc. of the metal). Concentrations of the reactants, (A) $2{\cdot}00 \times 10^{-4}$ M, (B) $1{\cdot}33 \times 10^{-4}$ M, and (C) $1{\cdot}00 \times 10^{-4}$ M.

An inspection of Fig. 2 shows that the combining metal, ligand ratio is 1 : 3 suggesting the formation of a chelate of the type Co $(NRS)_3$.

For the evalution of the stability constant employing the present method, the absorbance of the mixtures determined at 420 mμ was plotted against the ratio Co^{++}/Co^{++}+NRS (Figure 3).

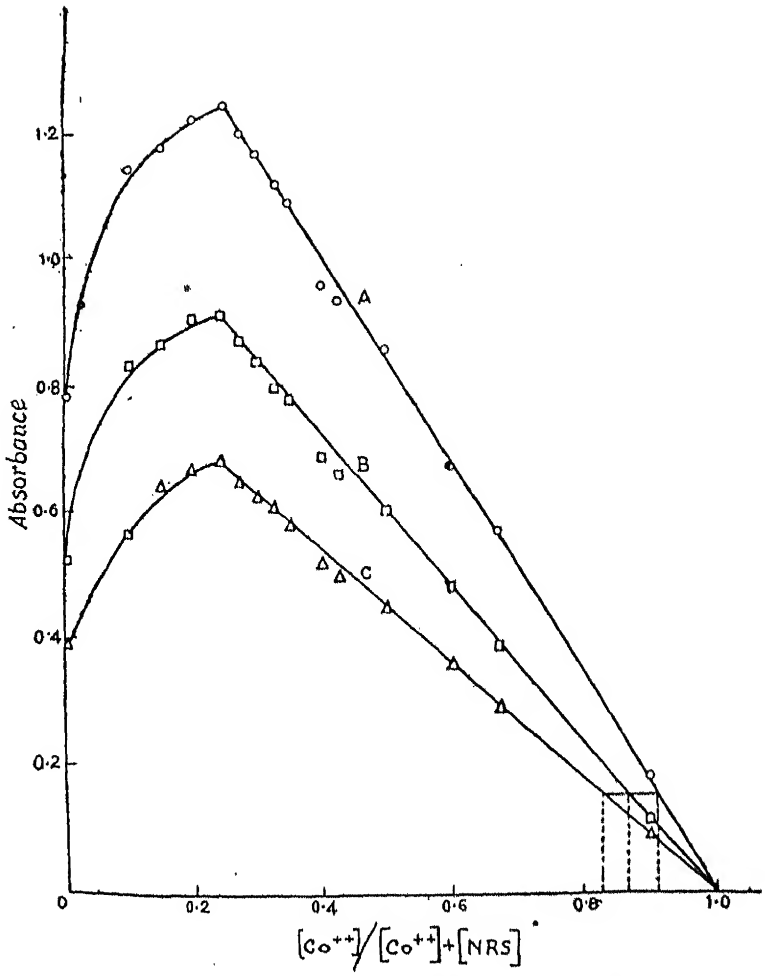

Fig. 3. Determination of the Stability Constant. Concentration of the reactants, (A) 2·0 × 10^{-4} M, (B) 1·33 × 10^{-4} M and (C) 1·00 × 10^{-4} M.

An inspection of Fig. 3 again shows that the stoichiometric breaks in the curve occur at the metal, ligand ratio of 1 : 3.

DISCUSSION

In the system investigated in the present work, the metal ion at concentrations employed are colourless and therefore, with progressive increase of M, Ke decreases and in the descending portions of the curve it may be assumed that the majority of the chelating agent is chelated with the metal ion contributing no appreciable absorbance to the absorbance of the chelate *i.e.* the descending parts of the curve represent the absorbance value of the chelate alone. It may, therefore, be assumed that in curves A, B and C in figure 3, where the absorbance are the same, the respective amounts of the chelate formed in each case are identical.

In the system under investigation, three equimolar concentrations heve been employed for the continuous variations data. From Fig. 3, two sets of values of a_1, b_1 ; a_2, b_2 where the absorbances *viz*. 0·15 is the same in each case have been obtained. The value of x was then calculated employing equation (6) from which the values of the stability constant were finally obtained by the use of the equation (1). The results are represented in Table 1.

TABLE 1

Curves	Sets of values of a_1, a_2 ; b_1 and b_2 Extrapolated from Fig. 3	x obtained from equation (6)	log K calculated from equation (1)
A and B	$a_1 = 1{\cdot}82 \times 10^{-4}$ $a_2 = 1{\cdot}16 \times 10^{-4}$ $b_1 = 0{\cdot}18 \times 10^{-4}$ $b_2 = 0{\cdot}17 \times 10^{-4}$	$28{\cdot}82 \times 10^{-7}$	13·29
B and C	$a_1 = 1{\cdot}16 \times 10^{-4}$ $a_2 = 0{\cdot}83 \times 10^{-4}$ $b_1 = 0{\cdot}17 \times 10^{-4}$ $b_2 = 0{\cdot}17 \times 10^{-4}$	$2{\cdot}2293 \times 10^{-6}$	13·27

Average value of
log K = 13·28

The above value of log K obtained by the continuous variation method has been correlated with that obtained by the mole ratio[7] method and with the values of the free energy change (ΔG^o) and presented in the table below :

TABLE 2

System	pH	log K (25°C)	ΔG^o at 25°C (K. cals)	
Co^{++}–NRS	6·5	13·28	–18·23	(*i*)
		13·5	–18·52	(*ii*)

(*i*) the present method (*ii*) the mole ratio method.

The method discussed above for the evalution of the stability constant of the complexes of the type 1 : 3 thus gives results which are fairly accurate and reproducible.

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REFERENCES

1. Job, P *Ann. Chim.*, **10** (9) : 113, (1928).
2. Vosburgh, W. C. and Cooper, G R. *J. Amer. Chem. Soc.*, **63** : 437, (1941).
3. Katzin, L. J. and Gebert, E *J. Amer. Chem Soc.*. **72** : 5455, (1950).
4. Mushran S. P. and Coworkers. *J. Ind. Chem. Soc.*, **43** : 4, (1966)
5. Foley, R. T and Anderson, R. C. *J. Amer. Chem. Soc.*, **70** : 1195, (1948) ; **71** : 909, (1949).
6. Dey, A. K. and Coworkers. *J. Inorg. Nucl. Chem*, **6** : 314 (1958) ; *Anal. Chim Acta.*, **18** : 324, (1958).
7. Yoe, J. H. and Jones, A. L. *Ind Eng. Chem. Anal. Ed.*, **16** : 111, (1944).

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